ANALYTIC GEOMETRY

A FIRST COURSE

ANALYTIC GEOMETRY

A FIRST COURSE

BY

WILLIAM H. MALTBIE

1906

THE SUN JOB PRINTING OFFICE

BALTIMORE

TABLE OF CONTENTS

Table of Contents.

Table of Contents.

ANALYTIC GEOMETRY.

CHAPTER I.

INTRODUCTORY.

1. The student who begins the study of Analytic Geometry is assumed to have acquired previously a knowledge of elementary geometry and of algebra.

In geometry he has used as subject matter the geometric elements (point, line, plane, etc.) and, starting from certain well defined axioms and postulates, has deduced the properties of the simpler plane and solid geometric forms. He has also developed a method of investigation which enables him to deal with questions of form, and of magnitude so far as it depends on form. In algebra he has used as subject matter certain symbols, some quantitative ($x, y, a, \ldots$), some operational ($+, -, =, \surd, \div, \ldots$); and has developed a method, far more general than that of arithmetic, of dealing with questions of quantity.

Each of these methods of mathematical investigation (geometric and algebraic) has its advantages and its limitations. In geometry we work with elements which actually possess the properties of form, position, and magnitude in which we are interested, and the method of proof is in consequence frequently suggested by a glance at the diagram. In algebra we work with symbols which have in themselves no properties and carry no power of suggestion. Consider the two statements

a. Two straight lines cannot intersect in more than one point;

b. The equations $2x - 3y + 1 = 0$
and $4x + 3y - 3 = 0$
cannot be simultaneously satisfied for more than one pair of values of the variables.

These statements as we shall see later are practically one and the same; but the first is evident as soon as the diagram is drawn, while the second requires a proof of whose form the equations themselves give no hint. The advantage of the geometric method is evident.

The algebraic method on the other hand possesses a great advantage over the geometric in the remarkable generality of its processes and results. It includes whole classes of problems in a single equation and expresses the solution of them all in a single formula. For example all quadratics may be included in the single form

$$ax^2 + 2bx + c = 0,$$

and the solutions of them all are embraced in a single formula,

$$x = -\frac{b}{a} \pm \frac{1}{a}\sqrt{b^2 - ac}.$$

2. Analytic Geometry is an attempt to establish a relation between these two branches of mathematics, so that the methods of either may be applied to the other; in short, an attempt to establish such a connection that one may write the formula of a curve or draw the diagram of an equation.

At the very outset we are confronted by a difficulty. Geometry deals almost wholly with fixed objects, definite points, lines, and planes, while algebra is concerned largely with variables. But we can obviate this difficulty by thinking from now on of all curves as traced out by a variable point, whose variation in position we shall attempt to connect with the change in value of algebraic variables.

CHAPTER II.

ANALYTIC GEOMETRY OF ONE DIMENSION.

3. THE SYSTEM OF CO-ORDINATES.

Consider the simplest case of algebraic variation. A single variable x is free to take all possible values from minus infinity to plus infinity. An equation of the first degree,

$$x - a = 0,$$

stops the variation of x and compels it to take the value a. An equation of the second degree,

$$(x - a)(x - b) \equiv x^2 - (a + b)x + ab = 0,^*$$

compels x to take one of the two values a or b; and similarly for the equations of the higher degrees. Compare all this with the simple case of geometric variation by which a point traces out a straight line. The point takes an infinite number of different positions, between which and the values of x we may establish a one to one correspondence. But in order to do this we must make certain purely arbitrary assumptions.

A. We must assume that a certain point on the line is to represent the zero value of x. This point we shall call the origin. Let it be the point 0.

FIG. 1.

B. In order to represent both the positive and negative values of x we must divide our line into positive and negative portions. We assume that all points on the right of the origin correspond to positive values of x, and all points on the left to negative values.

C. We must decide upon a scale of measurement; i. e., whether we will measure distances on the line in feet, inches, millimeters, or by some other arbitrary scale.

---

*The symbol $\equiv$ is used to denote an identity. The student must be thoroughly familiar with the distinction between conditional and identical equations.

Having made these three assumptions, we agree that the point at the distance a (measured by the agreed upon scale) from the origin shall correspond to the value a of x, and the desired correspondence between the point and the variable is now established. As x changes continuously from minus infinity to plus infinity, the point, by virtue of the assumptions we have made, is forced to move continuously from one end of the line to the other, while to any one value of x there corresponds one and only one position of the point and conversely.

The distance of a point from the origin is called the co-ordinate of the point, while the line which is used as the basis of the system is called the axis of co-ordinates, or merely the axis.

4. THE RELATION OF THE POINT AND THE EQUATION. If a point is at the distance a from the origin the equation

$$x = a, \text{ or } x - a = 0,$$

is called the equation of that point. Thus if the centimeter is the unit of measure the point P in Fig. 1 is spoken of as the point 4, and its equation is

$$x - 4 = 0.$$

It is at once evident that every first degree equation in one variable represents a single point, and conversely that any point on the line may be represented by a first degree equation in one variable.

The second degree equation in one variable can be written as the product of two factors, and is satisfied when either of the two factors is equal to zero. It is accordingly said to represent the two points which would be represented by the two factors taken singly. For example,

$$x^2 - 3x + 2 \equiv (x - 2)(x - 1) = 0$$

represents the two points at distances two and one to the right of the origin.

PROBLEMS.

Locate the points represented by the following equations:

1. $3x = 4$
2. $x + \frac{3x}{2} = 5$
3. $\frac{y}{2} = \frac{2}{3}$
4. $t - \frac{3}{2} = \frac{7}{2} - 10$
5. $x^2 - 5x + 6 = 0$
6. $2x^2 - 7x - 2 = 0$
7. $x(x^2 + 4x - 12) = 0$
8. $(2x^2 - 10x - 12)(x^2 - 1) = 0$
9. $x^2 - 1 = 7x - 3$
10. $x^3 - 4x = 0$

11. Write the equation which represents two points on opposite sides of the origin and at a distance five from it.

12. If the unit of length adopted be the foot, write the equations of the points whose distances from the origin are respectively 3 inches, 3 feet, 3 yards, minus 1 yard.

13. Write the equation representing the three points whose co-ordinates are $a, \frac{1}{2}, \frac{4}{3}$.

14. Generalize the work in problem 11; i. e., write the equation which, if proper values are given to the constants involved, will represent any pair of points symmetrically situated with respect to the origin.

15. Write the general equation representing a triad of points whose co-ordinates are in the ratio of 2, 3, 4.

5. THE EQUATION WITH EQUAL ROOTS.

The equation

$$x^2 - 2ax + a^2 = 0$$

demands special consideration. It reduces at once to

$$(x - a)(x - a) \equiv (x - a)^2 = 0,$$

and we might say that the equation represents only the point at the distance a from the origin. But this mode of interpretation is unwise because it fails to recognize any distinction between the two equations

$$x^2 - 2ax + a^2 = 0$$

and

$$x - a = 0.$$

We shall obtain a hint of the proper interpretation if we ask how the equation under consideration arises. It is evidently the limiting form of

$$(x - a)(x - b) \equiv x^2 - (a + b)x + ab = 0$$

as b tends to a as a limit. We may therefore say,

"$x^2 - 2ax + a^2 = 0$

is the limiting form of an equation representing two distinct points as the points tend to coincidence." Mathematicians, however, are accustomed to use the shorter expression,

"$x^2 - 2ax + a^2 = 0$

represents two coincident points." understanding by this exactly what is expressed in the longer phrase above.

The method of interpretation adopted here for this exceptional case of the second degree equation is a general method, widely used in all branches of mathematics, and the student should obtain a clear understanding of it. It may be stated as follows:

1. Exceptional cases will be treated by regarding them as limiting forms of the more general case. 2. If any expression has been used to represent the more general case, the limiting form of that expression will be used, whenever it is possible, to represent the special case.*

6. THE EQUATION WITH COMPLEX ROOTS.† Another type of equation presents a more serious difficulty. When $b^2 - ac$ is negative the equation

$$ax^2 + 2bx + c = 0$$

is not satisfied by any real value of x. Consider the special case of

$$x^2 - 2x + 2 = 0.$$

Here

$$x = 1 \pm \sqrt{-1}$$

and there are no points on the line corresponding to these values of x. The difficulty grows out of the nature of our fundamental assumptions. We have established such a connection between the points on the line and the values of x that every point corresponds to a real value of x and there are no points left to correspond to imaginary or complex values. We shall accordingly say that the equation represents a pair of imaginary points, meaning thereby merely that it represents a pair of points that from the nature of our fundamental assumptions cannot be represented in our diagram.

*As another illustration consider the treatment of the special case of division by zero. It is regarded as the limiting form of division by a quantity b, as b tends to zero. It is evident, provided the dividend is not zero, that as the divisor tends to zero the quotient increases indefinitely. Our general expression for division is $\frac{a}{b} = c$, and in accordance with the second part of our principle we write $\frac{a}{0} = \infty$. This expression must not, however, be regarded as containing any statement as to the possibility of dividing something by nothing. It is merely a short hand way of saying, "When the dividend is not zero and the divisor tends to zero as a limit the quotient increases indefinitely." Other examples of this mode of interpretation will be met with from time to time in our work.

†A complex number is one of the form $a + b\sqrt{-1}$. When a is zero the number is a pure imaginary, when b is zero it is real. From now on we shall denote $\sqrt{-1}$ by i and the general form of a complex number will be $a + ib$.

PROBLEMS.

What points do the following equations represent?

1. $x^2-3x-28=0$
2. $3x^2-2x+1=0$
3. $(x^2-4x+9)(x-2)=0$
4. $(x^2-1)(x^2+1)=0$
5. $x^2(x^2-3x-\frac{5}{4})=0$
6. $x(x^2-4x+1)(x^2+2)=0.$

7. TRANSFORMATION OF CO-ORDINATES IN ONE DIMENSION. In building our system of co-ordinates we made three purely arbitrary assumptions: first, that the origin was at a particular point; second, that distances to the right were to be counted positive; third, that distances were to be measured by a given scale. A change in any one of these assumptions will of course lead to a new system of co-ordinates, and the algebraic relation between the two systems must be determined.

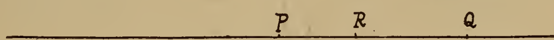

FIG. 2.

Suppose for example that the point P has been taken as the origin, distances to the right as positive, and $\frac{1}{10}$ of an inch as the unit of measurement. The point Q will then have the co-ordinate 10 and be represented by the equation

$$x-10=0.$$

If now we take as a new origin a point R at a distance 4 to the right of P, and, to avoid confusion, call the variable in this new system x', we see at once that the old and new co-ordinates of any point are connected by the relation

$$x=x'+4,$$

and that the new equation of Q is therefore

$$x'-6=0.$$

In general a movement of the origin a distance h, positive or negative, corresponds to the algebraic substitution

$$x=x'+h.$$

If we change our assumption as to the directions and make distances to the left positive, it is evident that the corresponding substitution is

$$x=-x'.$$

If we change our assumption as to the unit of length, and replace it by one k times as great, it is evident that the corresponding algebraic substitution is

$$x = kx'.$$

If we note that the substitution

$$x = x' + h$$

followed by $\qquad x' = -x''$

followed by $\qquad x'' = kx'''$

gives us $\qquad x = -kx''' + h.$

we see that the substitution

$$x = ax' + b$$

may be made by proper choice of the constants a and b to represent any change whatever of the system of co-ordinates. Any such change is called a transformation of co-ordinates and the corresponding algebraic substitution is called a formula of transformation.

The mathematical importance of the subject of transformation of co-ordinates grows out of the power which the corresponding substitutions give us to modify equations. For example, the substitution

$$x = x' + 2$$

reduces $\qquad x^2 - 4x + 3 = 0$

to the simpler form

$$x'^2 - 1 = 0,$$

while the substitution

$$x = x' + 1$$

reduces the same equation to

$$x'^2 - 2x' = 0.$$

Again the equation

$$9x^2 - 6x - 8 = 0$$

which has the fractional roots $\frac{4}{3}$ and $-\frac{2}{3}$ is reduced by the

substitution $\qquad x = \frac{x'}{3}$

to the form $\qquad x'^2 + 2x' - 8 = 0$

which has the integer roots 4 and 2.

PROBLEMS.

Explain the geometric significance of the following transformations.

1. $x = x' + 2$ 2. $x = x' - 4$
3. $2x = 2x' + 3$ 4. $x = -2x'$
5. $x = -3x' - 4$ 6. $2x = 5 - 6x'$

Write the formulae of transformation which shall represent the following changes of the system of co-ordinates:

7. A movement of the origin a distance 4 to the left.

8. A movement of the origin to the point whose co-ordinate is $-\frac{3}{2}$.

9. A division of the unit of length by 4.

10. A movement of the origin a distance 2 to the right, followed by an interchange of the positive and negative portions of the axis.

11. A movement of the origin to the point whose co-ordinate is 4, followed by a multiplication of the unit of length by three, followed by an interchange of the positive and negative portions of the axis.

8. THE DERIVATION OF THE FORMULA OF TRANSFORMATION WHICH WILL PRODUCE A GIVEN RESULT.

We are not always given the formula of transformation. Cases frequently arise in which we are required to determine the substitution which will produce a given form when applied to a particular equation. For example, let it be required to find the transformation that will reduce the equation

$$x^2 - 5x + 6 = 0$$

to a new equation which has no term of the first degree in the variable. Two methods of solution present themselves. We may find the two points represented by the equation and then accomplish the desired result by taking the point midway between them as the new origin, (Prob. 14, Art. 4); or we may assume a general substitution

$$x = x' + h,$$

and note that the resulting equation

$$x'^2 + x'(2h - 5) + h^2 - 5h + 6 = 0$$

has no term of the first degree in the variable if

$$2h - 5 = 0, \text{ i. e., if } h = \frac{5}{2}$$

Our substitution is therefore determined.

The second of the methods used is the more general and therefore of course the more valuable.

PROBLEMS.

Use each of the methods outlined above to determine the substitutions which will reduce the following equations to equations having no term of the first degree in the variable:

1. $x^2+3x-7=0$ 2. $x^2-4x-1=0$
3. $x^2-5x+2=0$ 4. $x^2-2x+1=0$.
5. Discuss in full the substitution which will reduce an equation of the second degree to an equation which has no constant term.
6. Show that no movement of the origin can change the degree of the equation.

9. DISTANCE RATIOS AND ANHARMONIC RATIOS.

If we are given two points A and B, whose co-ordinates are x_1 and x_2, and a third point C, whose co-ordinate is x_3, on the same line, the point C is said to divide the segment AB in the ratio

$$\frac{(x_1-x_3)}{(x_3-x_2)},$$

which is called the distance ratio of C with respect to A and B. This distance ratio is evidently numerically equal to the ratio of the segments $\frac{AC}{BC}$ and is positive or negative according as the point C is within or without the segment AB. In this latter fact lies its superiority to the mere numerical ratio, which makes no distinction between internal and external division.

If a fourth point D whose co-ordinate is x_4 is given, the ratio of the two distance ratios so determined

$$\frac{\dfrac{x_1-x_3}{x_3-x_2}}{\dfrac{x_1-x_4}{x_4-x_2}}$$

i. e.,

$$\frac{(x_1-x_3)(x_4-x_2)}{(x_1-x_4)(x_3-x_2)}$$

is called the cross ratio or the anharmonic ratio of the four points, and is denoted by

$$\{\ x_1\ x_2\ x_3\ x_4\ \}$$

If C and D divide the segment AB internally and externally in the same ratio, the anharmonic ratio is -1, the division is said to be harmonic, and C and D are said to be harmonic conjugates with respect to A and B.

PROBLEMS.

1. Find the distance ratio of the point 4 with respect to the points 3 and 7.
2. Find the anharmonic ratio of the points 2 and -1 with respect to the points 3 and 5.
3. Find the point whose distance ratio with respect to the points 2 and 4 is -5.
4. Find a point x such that the anharmonic ratio of 7 and x with respect to 4 and 5 shall be 2.
5. Find the harmonic conjugate of the origin with respect to 4 and -1.
6. Generalize problem 3, i. e., find the point whose distance ratio with respect to x_1 and x_2 shall be k.
7. Show that the definition given above of harmonic division is equivalent to the following: The segment AB is divided harmonically by P and Q if $\frac{1}{QA}+\frac{1}{QA}=\frac{2}{QP}$.

CHAPTER III.

SOME FUNDAMENTAL IDEAS OF THE ANALYTIC GEOMETRY OF TWO DIMENSIONS.

10. **THE SYSTEM OF CO-ORDINATES.**

We have built up in the preceding pages an analytic geometry of one dimension in order to illustrate the variation of a single algebraic variable. Let us now consider the less simple case of two variables.

Two variables, x and y, are each free to take any value from minus infinity to plus infinity and these values are paired (one of x with one of y) in all possible ways. We desire to establish a one to one correspondence between these pairs of values and the members of some group of geometric objects, just as we established a one to one correspondence between the values of one variable and the points of a straight line. Since with any one of the infinite number of values of x may be paired any one of the infinite number of values of y, the total number of pairs is infinity squared, or to use a more satisfactory phrase, is doubly infinite, or an infinity of the second order.* The number of points in a line however is singly infinite, and our former mode of representation therefore fails us, since we cannot establish a one to one correspondence between a singly infinite number of points and a doubly infinite number of pairs of values. The plane however contains a doubly infinite number of points and may therefore be used.

In order to establish this correspondence between the pairs of values of x and y and the points of the plane we must make certain arbitrary assumptions.

---

*See Appendix A. Infinities of various orders.

A. We assume two intersecting straight lines as a basis of reference. Let them be OA and OB.

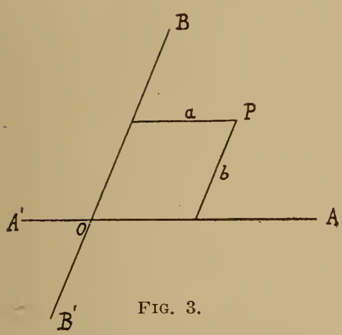

FIG. 3.

B. We assume positive and negative directions on these two lines. Let OA and OB be positive, and OA' and OB' negative.

C. We assume that values of x correspond to distances from the line BB' measured (by any desired unit of length) parallel to the line AA'.

D. We assume that values of y correspond to distances from the line AA' measured (by any desired unit of length) parallel to the line BB'. (The units determined by the assumptions C and D will usually but not always be the same.)

The lines AA' and BB' are called Axes of Co-ordinates. AA' is the X axis or axis of abscissas. BB' is the Y axis or axis of ordinates. The intersection of the axes is the origin. The distance of a point from the Y axis, measured parallel to the X axis, is called the abscissa, or the x co-ordinate, or frequently merely the x of the point. The distance of a point from the X axis, measured parallel to the Y axis, is called the ordinate, or the y co-ordinate, or frequently merely the y of the point. Thus in the diagram the point P has the abscissa a, and the ordinate b. The point P is frequently spoken of as the point (a, b). In this notation the x co-ordinate is always the one first mentioned. When the axes are perpendicular to each other the co-ordinates are called rectangular co-ordinates, otherwise they are oblique co-ordinates. From now on we shall understand rectangular co-ordinates to be used unless it is otherwise stated.*

The four assumptions made above establish a complete correspondence between the points in the plane and all pairs of real values of x and y. To each point corresponds one and only one pair of values and conversely.

*Such a system of co-ordinates as we have outlined above is frequently called Cartesian, in honor of Descartes, who was the first worker in this field. Consult Ball's Short History of Mathematics or some similar work.

PROBLEMS.

1. Locate the following points: (1, 1), (—1, —1), (0, 3), (2, 0), (—5, 3), (0, 0), (5, —1), $\left(-2, 4\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{3}\right)$, (—2, —4), (0, —5).

2. The lower left hand vertex of a square whose side is 4 is at the origin and two sides coincide with the axes. Find the co-ordinates of the other three vertices.

3. The upper right hand vertex of a rectangle whose sides are 8 and 3 is at the origin and its longer side coincides with the X axis. Find the co-ordinates of the middle points of the sides.

4. The base of an equilateral triangle whose side is 4 is parallel to the X axis, and the point (0, 2) is the middle point of the base. Find the co-ordinates of the three vertices.

5. The axes are the diagonals of a square whose side is 8. Find the co-ordinates of the vertices.

6. A regular hexagon whose side is 5 has its center at the origin and one pair of vertices on the X axis. Find the co-ordinates of the remaining vertices.

11. THE SIGNIFICANCE OF TWO SIMULTANEOUS EQUATIONS. If x and y are subject to no restriction the corresponding point may take any position in the plane. If, however, we are given two equations which the variables must satisfy, the point is no longer free. For example

$$3x - y - 5 = 0$$

and

$$5x + y - 11 = 0$$

are both satisfied only when we have

$$x = 2,\ y = 1.$$

These two equations therefore restrict the varying point to the single position (2, 1), and may therefore be said to represent this point. Again

$$x^2 + y^2 - 6x + 4 = 0$$

and

$$x + y = 0$$

may be said to represent the two points (1, 1) and (2, 2) since only for these values of x and y can these two equations be simultaneously satisfied.

Since two algebraic equations in two variables always on solution give a finite number of pairs of values of the variables, it follows that two algebraic equations in two variables restrict the varying point to a finite number of fixed positions.

PROBLEMS.

Find the points represented by the following pairs of simultaneous equations:

1. $4x - y + 22 = 0$
   $x - 2y + 9 = 0$
2. $3x + y + 2 = 0$
   $6x - 3y - 11 = 0$
3. $4x - y - 4 = 0$
   $5x + 2y - 5 = 0$
4. $x + 2y = 0$
   $3x - 4y = 0$
5. $x^2 + y^2 - 9 = 0$
   $x + 2y - 1 = 0$
6. $y^2 - 8x = 0$
   $x + 2y = 0$

12. THE SIGNIFICANCE OF A SINGLE EQUATION. LOCI.

When we have a single equation the values of x and y are no longer determined, but are nevertheless not entirely free. The equation acts as a restriction on the movement of the variable point, without fixing its position. For example

$$x - y = 0$$

is satisfied by the points* $(1, 1)$, $(2, 2)$, $(3, 3)$ but not by the points $(1, 2)$, $(1, 4)$, $(2, 3)$. In fact it is satisfied by every point whose co-ordinates are equal and of the same sign and by no others. It is therefore satisfied by all points on that bisector of the angle between the axis which passes through the first and third quadrants, and by no others. This bisector is therefore called the locus of points subject to the given condition, and

$$x - y = 0$$

is called the equation of the bisector, or the equation of the locus. In the same way any equation between x and y acts as a restriction on the variation of the point which has x and y as its co-ordinates. The aggregate of all points which satisfy a given condition is said to constitute a locus, and the equation which expresses the condition is called the equation of the locus. The student may accept without proof for the present the statement that the locus, as used in analytic geometry of two dimensions, consists in general of one or more lines, straight or curved.

It sometimes happens that the left hand member of the given equation is the product of two or more rational factors. In this case it is evident that the equation will be satisfied by such

---

*This is the usual abbreviated form for the expression, "The equation $x - y = 0$ is satisfied by the co-ordinates of the points,' etc.

points as make any one of the factors equal to zero; i. e., the locus in this case consists of two or more parts, each of which would be represented by equating one of the factors to zero. Such a locus is said to be a degenerate locus. For example

$$(x-y)(x+y)=0$$

is satisfied by all points on the bisector through the first and third quadrants, but also by the points on the bisector through the second and fourth quadrants. The equation is said therefore to represent the degenerate locus consisting of these two bisectors.

This relation between equation and locus is the fundamental idea in our present subject. The problems to which it gives rise may usually be divided into the following groups:

a. To construct the locus when the equation is given.

b. To construct the locus when a finite number of points upon it is given.

c. To deduce the equation when the restrictions on the movement of the tracing point are given.

d. To deduce the equation when a finite number of points on a locus of some known type is given.

e. To deduce the geometric properties and relations of loci from a consideration of the corresponding equations.

These problems we shall now proceed to treat in turn.

CHAPTER IV.

TO CONSTRUCT THE LOCUS WHEN THE EQUATION IS GIVEN.

13. THE PROCESS OF CURVE PLOTTING. It sometimes happens, as in the case discussed at the opening of the previous paragraph, that the equation is so simple that the form of the locus can be at once inferred from the equation; and the ability to recognize in this way a large number of simple loci can easily be acquired.

In more complex cases it is not so easy to infer the form of the locus from the equation, and the following method is adopted. Assume in succession a number of values of one variable, substitute these in the equation and compute the corresponding values of the other. Each of the pairs of values thus determined will locate a point on the locus. By locating a sufficient number of such points the form of the locus may be inferred. For example, consider the equation

$$x - 3y + 2 = 0.$$

If we substitute in succession for x in this equation the integer values from -7 to $+7$, and compute the corresponding values of y, we obtain the following points which satisfy the equation:

$$(-7, -\tfrac{5}{3}),\ (-6, -\tfrac{4}{3}),\ (-5, -1),\ (-4, -\tfrac{2}{3}),\ (-3, -\tfrac{1}{3}),$$
$$(-2, 0),\ (-1, \tfrac{1}{3}),\ (0, \tfrac{2}{3}),\ (1, 1),\ (2, \tfrac{4}{3}),\ (3, \tfrac{5}{3}),\ (4, 2),$$
$$(5, \tfrac{7}{3}),\ (6, \tfrac{8}{3}),\ (7, 3).$$

If we locate these points on our diagram we shall find that they all lie on a straight line, and consequently we may infer that this line is the locus. But in drawing this inference we make two assumptions: first, that the point which traces the locus moves continuously and not by leaps from point to point, or in other

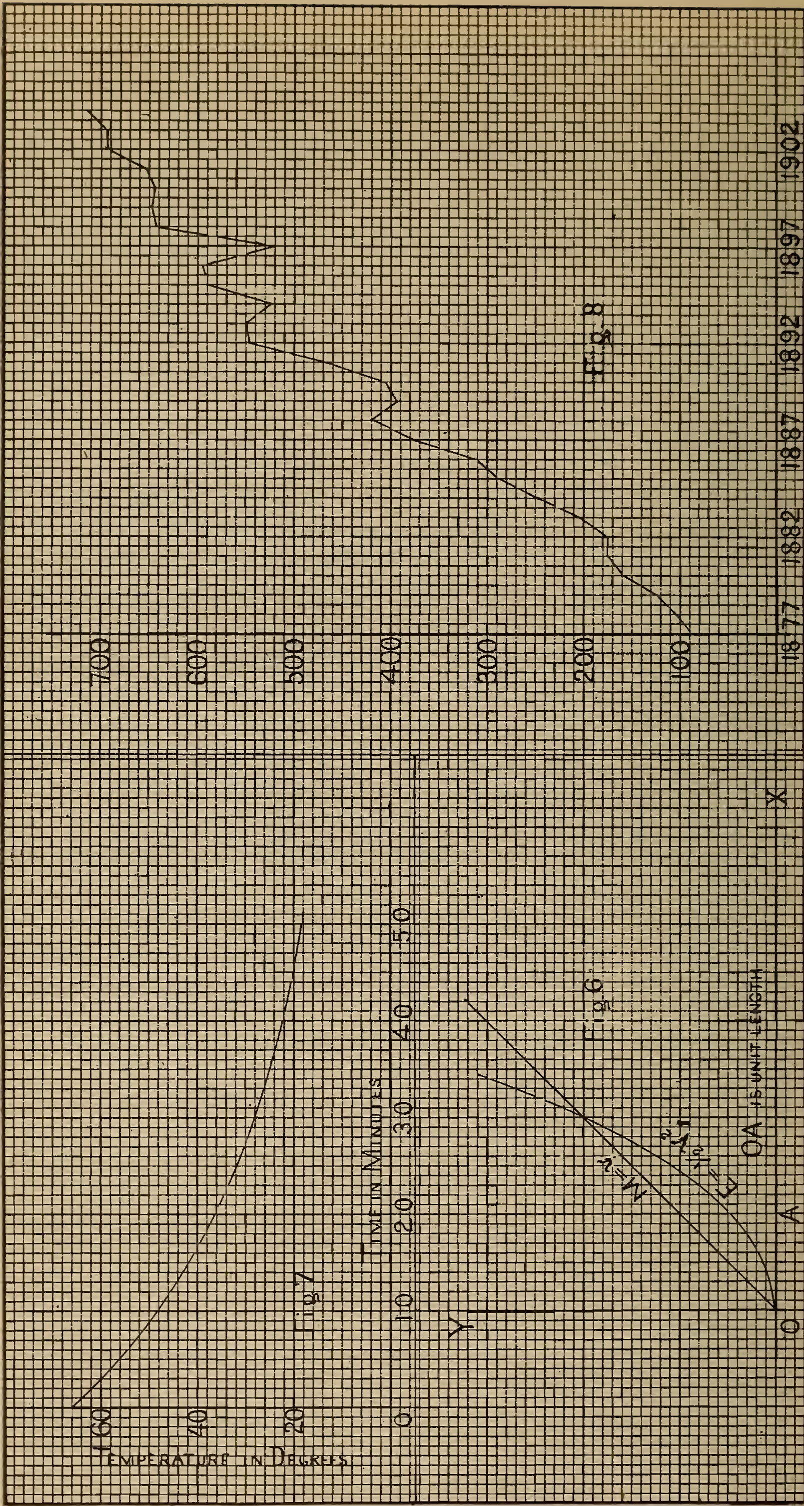
Fig 6
OA is unit length
O
A
X
Y
Fig 7
Time in Minutes
0
10
20
30
40
50
Temperature in Degrees
20
40
60
Fig 8
1877
1882
1887
1892
1897
1902
100
200
300
400
500
600
700

words that the locus is a continuous curve and not a number of separate fragments or isolated points; second, that the tracing point moves along the curve AB rather than along some other curve through the same points, as the undulating curve in the figure.

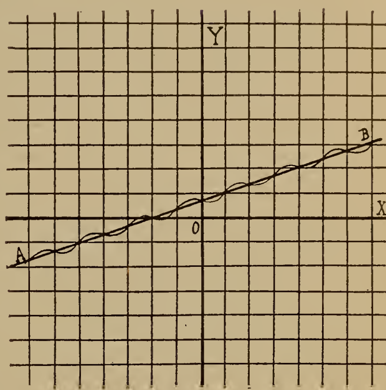

FIG. 4.

We can practically convince ourselves of our right to make these assumptions in this particular case by taking values of x between those already taken, and in this way finding additional points of the locus. The mathematical treatment of these difficulties must be deferred to a later point in the student's career.

As a second example, consider the equation

$$4x^2+9y^2=36.$$

Solve the equation for y and we have

$$y=\pm\frac{2}{3}\sqrt{9-x^2}.$$

We see at once that it is useless to look for points whose x does not lie between -3 and $+3$, since any value of x outside these limits gives y imaginary values, and our fundamental assumptions are such that a pair of values, one or both of which are imaginary, has no corresponding point in the plane. Giving x a series of values between -3 and $+3$, and computing the corresponding values of y, we obtain the following points which satisfy the equation

$(-3, 0), (-2, \pm\frac{2}{3}\sqrt{5}), (-1, \pm\frac{4}{3}\sqrt{2}), (0, \pm 2), (1, \pm\frac{4}{3}\sqrt{2}),$

$(2, \pm\frac{2}{3}\sqrt{5}), (3, 0).$

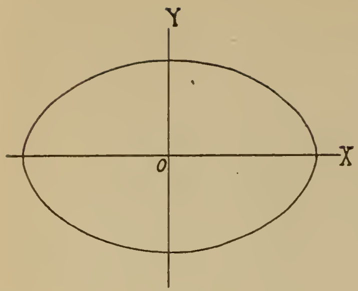

FIG. 5.

Plotting these points we readily infer the curve to be of the form in the figure, an inference which we may confirm as in the last example.

It is sometimes more convenient to assume arbitrary values for y and compute the corresponding values of x, as for example in the equation

$$x = y^2 - 3y + 2.$$

The variable to which arbitrary values are assigned is called the independent variable, the other the dependent variable. The distinction is evidently a purely arbitrary one and the student will in each case so choose the independent variable as to make his work as simple as possible.

14. UTILITY OF THE PROCESS. The process illustrated in the last paragraph, and usually spoken of as curve plotting, frequently enables us to visualize a formula and obtain a clearer idea of its significance than we could obtain in any other manner. For example, consider the case of a body of mass m moving with a velocity v. Its momentum M and its kinetic energy E are given by the formulae

$$M = mv$$

and

$$E = \frac{1}{2} mv^2.$$

A study of these formulae will give the student some idea of the relative variation of momentum and energy as the velocity changes, but a much clearer idea will be obtained by an examination of Fig. 6, where the corresponding curves are plotted on the same axes. In each case the horizontal axis is chosen as the axis of velocities, and in order to make the problem a definite one the mass m is taken as unity.

PROBLEMS.

Plot the following curves, inferring, whenever you can, the form of the curve directly from the equation.*

1. $x = 0$
2. $y = 0$
3. $y = 2$
4. $3x = 6$
5. $x + y = 0$
6. $x - 2y = 0$
7. $2x + 4y = 0$
8. $x^2 + y^2 = 4$
9. $x^2 + 2y = 0$
10. $x^2 + y^2 - 2x = 0$
11. $x + y - 1 = 0$
12. $y = x^2 + 3x - 1$

*The student should use co-ordinate paper ruled to tenths.

13. The distance passed over by a moving body in t seconds is given by the formula

$$d = v_0 t + \frac{1}{2} a t^2,$$

where v_0 is the initial velocity and a is the acceleration. Plot the curves for the three special cases

(1) $v_0 = 0, \ a = 2,$
(2) $v_0 = 2, \ a = 0,$
(3) $v_0 = 2, \ a = 2,$

and compare them.

14. The intensity of the light at any point varies inversely as the square of the distance of that point from the source of light; i. e., $i = \frac{k}{d^2}$. Let k equal one, and plot the corresponding curve.

15. The intensity of the magnetic field in the vicinity of a wire carrying an electric current varies inversely as the distance from the wire; i. e., $i = \frac{k}{d}$. Let $k = 1$, plot the corresponding curve, and compare it with the results of the last problem.

16. Show that the locus represented by the equation

$$ax + by + c = 0$$

can have no point in the first quadrant if a, b, and c are all positive.

17. State a corresponding theorem for the third quadrant.

CHAPTER V.

TO CONSTRUCT THE LOCUS WHEN A FINITE NUMBER OF POINTS UPON IT IS GIVEN.

15. THE METHOD OF PLOTTING. The equation of the locus is not always given. The law which regulates the phenomena may not be known or may be too complex for simple mathematical expression. We may however still have sufficient data to plot the curve. For example, a student in the laboratory desires to study the rate at which a body cools. He may not know the formula which connects the temperature of the body with the length of time it has been allowed to cool; but he records the temperature at short intervals and secures in this way a number of pairs of values of time and temperature which he may regard as co-ordinates, and thus plot a number of points on the curve. This curve (Fig. 7, p. 20) shows clearly the relation between temperature and time of cooling.

Again, the number of students in a given institution from year to year depends on too many causes for the relation to be expressed by an equation. But the curve can be plotted and thus a clear and condensed representation of the variation can be secured. The attendance at Johns Hopkins University from 1877 to 1904 is given in the subjoined table.

1877	89	1884	249	1891	468	1898	641
1878	104	1885	290	1892	547	1899	649
1879	123	1886	314	1893	551	1900	645
1880	159	1887	378	1894	522	1901	651
1881	176	1888	420	1895	589	1902	694
1882	175	1889	394	1896	596	1903	695
1883	204	1890	404	1897	520	1904	715

A glance at the corresponding curve (Fig. 8) gives however a much clearer picture of the growth of the institution.

PROBLEMS.

1. The following table gives the number of years one may expect to live at the ages indicated:

Age	*Expectation*	*Age*	*Expectation*	*Age*	*Expectation*
0	39.9	35	29.4	70	8.4
5	49.1	40	26.0	75	6.4
10	47.0	45	22.7	80	4.9
15	43.1	50	19.5	85	3.7
20	39.4	55	16.4	90	2.8
25	36.1	60	13.5	95	2.1
30	32.7	65	10.8	100	1.6

Plot the corresponding curve.

2. The temperature of a fever patient was as follows:

July 8,	5	P. M.	99.4	July 13,	12.30	P. M.	106.3
July 9,	6	A. M.	98.0	July 13,	1.30	P. M.	105.6
July 9,	5	P. M.	105.0	July 13,	5	P. M.	104.6
July 10,	6	A. M.	99.0	July 14,	6	A. M.	98.2
July 10,	5	P. M.	99.2	July 14,	5	P. M.	98.6
July 11,	6	A. M.	98.2	July 15,	6	A. M.	98.0
July 11,	1	P. M.	106.0	July 15,	5	P. M.	99.0
July 11,	2	P. M.	104.0	July 16,	6	A. M.	98.0
July 11,	5	P. M.	103.6	July 16,	5	P. M.	98.4
July 12,	6	A. M.	98.0	July 17,	6	A. M.	98.0
July 12,	5	P. M.	98.4	July 17,	5	P. M.	98.6
July 13,	6	A. M.	99.0	July 18,	6	A. M.	98.0

Plot the corresponding curve. (In this problem it will evidently be wise to take as the axis of X the line corresponding to some high temperature, such as the normal temperature of 98 degrees, in place of the line corresponding to zero degrees.)

4. Select some prominent stock or article of produce and plot its prices for the next two weeks as given in the daily stock or market reports, giving the reasons for any important fluctuations.

16. CHOICE OF SCALES OF MEASUREMENT. In these examples there is no particular reason why the same scale should be used on both axes. The fact that a certain distance has been used to denote a year or a day is no reason why the same distance should be used to denote a dollar or a degree of temperature. Mathematicians are consequently accustomed to select such scales as are most convenient, and to indicate their choice by a foot

note as in Fig. 6, or by figures on the axes as in Figs. 7 and 8. The choice is usually so made as to make the important features of the curve prominent. Thus in problem 2 the physician will choose a small distance to represent the day and a much greater one to represent the degree; while the curve of problem 1 may be almost wholly deprived of interest by the choice of a large distance to represent the unit of age and a small one to represent the unit of expectation. It is sometimes necessary after a curve has been drawn to one scale to make magnified drawings of certain portions in order to examine more closely certain doubtful points.

17. DETERMINATION OF INTERMEDIATE VALUES. When both the assumptions made in paragraph 13 can be granted, pairs of values intermediate between those actually known can be determined with a close degree of approximation by measuring the co-ordinates of the corresponding point on the curve. Thus in problem 1 of paragraph 15 the expectation for any intermediate age can be determined by measuring the ordinate corresponding to the abscissa representing that age. But in many cases this method of procedure is not in order since the assumptions of paragraph 13 cannot be made. In the Johns Hopkins problem for example the attendance does not pass continuously from 520 to 641, taking all the intermediate values, but passes by leaps from one value to another. In other words the actual locus is not a continuous line as we have drawn it, but a succession of disconnected points.

The second problem of paragraph 15 is a case in which the second assumption of paragraph 13 cannot be granted. While it is true that the temperature varies continuously with the time, it is by no means to be expected that our curve, based upon observations made at intervals of several hours, shows all these variations. There may be and probably are intermediate variations of which we have no record. In such a case as this where the nature of the phenomena is not sufficiently well understood to enable us to deny the existence of these intermediate variations, as we might in the age-expectation problem, and where the laws regulating the phenomena do not admit of algebraic expression, there is of course no way of determining the number or location of such intermediate variations. When the locus is determined by an equation they may be determined by the aid of the differential calculus.

18. DEFINITION AND TEST OF CONTINUITY.

The question of continuity, raised incidentally in the last paragraph, demands careful consideration. In order that a curve may be continuous it is necessary that as x (the abscissa of the tracing point) varies continuously, y (the ordinate of the tracing point) shall also vary continuously; or, expressed algebraically, y is a continuous function of x when the change in y due to a change in x may be made as small as we please by taking the change in x small enough. At all points where this condition is satisfied y is said to be a continuous function of x, at any point where it is not satisfied, y is said to be discontinuous. The fever temperature problem affords us an example of a continuous function. If the change in the time be small enough the change in the patient's temperature will be as small as we please, while in the Johns Hopkins problem no shortening of the interval will make the difference between two successive values of y a small quantity.

Let us examine the continuity of a simple function,* say

$$y = 2x^2. \dagger$$

If x increases by an amount $\overline{\triangle x}$, or as we more frequently say, takes an increment $\overline{\triangle x}$, y takes an increment which we may call $\overline{\triangle y}$.

Then $\quad y = 2x^2$

$$y + \overline{\triangle y} = 2(x + \overline{\triangle x})^2 = 2x^2 + 4x\overline{\triangle x} + 2\,\overline{\triangle x}^2$$

therefore $$\overline{\triangle y} = 4x\overline{\triangle x} + 2\,\overline{\triangle x}^2.$$

So long as x remains finite, the right hand side of this equation tends to zero as $\overline{\triangle x}$ tends to zero, or in other words $\overline{\triangle y}$ may, by a proper choice of $\overline{\triangle x}$, be made as small as we please for all finite values of x. That is, y is a continuous function of x so long as x is finite.

---

*If the student is not familiar with the ideas and notation of mathematical functionality, he should at this time read Appendix B.

†The method here used is applicable to the most complex forms, but numerous algebraic difficulties are encountered in the attempt to employ it. The overcoming of these difficulties falls in the province of the differential calculus.

19. INFINITE DISCONTINUITY. The definition of continuity given above brings to light also another sort of discontinuity. If the student will plot the curve

$$y(x-1)=x$$

he will note that as x tends to unity, y passes beyond all limit (i. e., becomes infinite), but so long as x differs ever so little from unity y remains finite. That is, no matter how small the increment which carries x from its previous value to the value unity, y leaps from a finite to an infinite value. The function is accordingly said to have an infinite discontinuity, while the discontinuities previously discussed are called finite discontinuities. Such functions as the student will meet in the present work are, as may be shown by the calculus, free from finite discontinuities, and such infinite discontinuities as may occur can be detected by plotting the curve.*

PROBLEMS.

Find the values of x for which the following functions have infinite discontinuities.

1. $y=\dfrac{x-2}{x-3}$
2. $y=\dfrac{1}{(x-1)(x-2)}$
3. $y=\sec x$

*The Johns Hopkins problem and problem 4 of paragraph 15 furnish examples of finite discontinuities, but in all these cases the functional relation is not given, so that they constitute no exception to the statement made above. As an example of a function with a finite discontinuity consider the equation

$$y=\frac{1}{1+e^{\frac{1}{x}}}$$

As x (regarded as positive) tends to zero, y tends to zero; but as x (regarded as negative) tends to zero, y tends to unity. The corresponding locus has therefore a finite jump from unity to zero as it crosses the Y axis.

CHAPTER VI.

TO DEDUCE THE EQUATION WHEN THE RESTRICTIONS ON THE MOVEMENT OF THE TRACING POINT ARE GIVEN.

20.
THE GENERAL METHOD.

Problems of this type by no means always occur in the simple form indicated in the heading of this chapter. Very frequently we are confronted with a mere verbal description of a finished curve, containing apparently no reference to the method of its construction. But if the verbal description is complete it contains all the limitations on the movement of the tracing point. The method of meeting the problem is therefore always the same. Consider the curve as traced by a variable point; determine the law which regulates the movement of the point, (i. e., the condition which is satisfied by all points on the curve and by no others) ; state this condition in algebraic form, i. e., express it as a relation between the variable co-ordinates, x, y, of the variable point. We have then an equation of condition between x and y which is satisfied by the co-ordinates of all points on the curve and by no others; in other words we have the equation of the given curve.

The student may get a somewhat clearer idea of this process if he will consider the analogy between it and the work of translating from one language into another. In translation from English into German for example, the student must have not only a knowledge of the German words equivalent to the English words in the passage to be translated, he must also have a knowledge of the peculiar forms, the idiomatic constructions, of the two languages. Ordinarily he will first of all throw the English sentence into the German order and replace the English idiom by the corresponding German idiom, and then is ready for the actual work of translation. Now algebra is after all to a great degree merely a language, and an equation is a sentence. The equation of a locus is the statement in algebraic language of the conditions

under which the tracing point moves, and the deduction of such an equation is merely the translation of the ordinary English description of those conditions into algebraic language. For example consider the circle of radius 2 centered at the point (4, 3). The algebraic idiom requires first of all that the curve be described as the locus of a moving point, and we accordingly throw our description of the curve into the new form, "A variable point moves in such a way as to keep its distance from the point (4, 3) equal to 2." The phrase, "A variable point" translates at once into, "The point (x, y)", and if we knew an algebraic equivalent for "The distance from (x, y) to $(4, 3)$" we should at once equate it to 2 and have the equation of the locus.

Since the definitions of the simpler curves are largely stated in terms of distance and direction, it will be wise before we take up the work proper of this chapter to develop some fundamental formulae which will enable us to translate questions of distance and direction into algebraic language.

21. DISTANCE BETWEEN TWO POINTS IN TERMS OF THEIR CO-ORDINATES.

Consider any two points P_1 and P_2 whose co-ordinates are (x_1, y_1) and (x_2, y_2). Draw $P_1 P_2$. Draw $P_1 Q_1$ and $P_2 Q_2$ parallel to OY and P_1R parallel to OX. Let D be the required distance. Then we have by direct application of our geometry

$$D = P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{P_2R}^2}$$

But $\quad P_1R = Q_1Q_2 = x_2 - x_1$

and $\quad P_2R = P_2Q_2 - P_1Q_2$

$$= y_2 - y_1$$

therefore

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

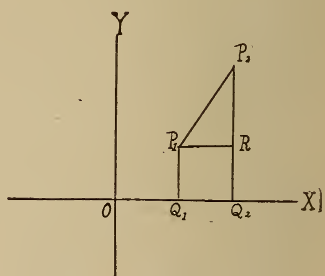

FIG. 9.

22. CO-ORDINATES OF A POINT WITH A GIVEN DISTANCE RATIO.

Given two points P_1 and P_2, to find the co-ordinates of a third point P_3 on the straight line joining P_1 and P_2 which shall have with respect to P_1 and P_2 a given distance ratio. (See paragraph 9.)

Let the co-ordinates of P_1, P_2, P_3, be (x_1, y_1), (x_2, y_2), (x_3, y_3) and let the numerical value of $\frac{P_1P_3}{P_2P_3}$ be $\frac{\lambda_1}{\lambda_2}$. The distance ratio P_3 with respect to P_1 and P_2 will then be either $\frac{\lambda_1}{\lambda_2}$ or $-\frac{\lambda_1}{\lambda_2}$ according as the point P_3 is within or without the segment P_1P_2.

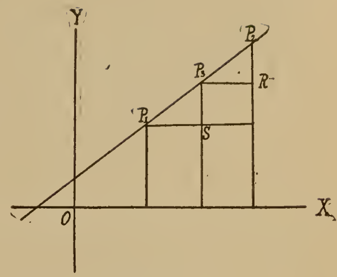

FIG. 10.

Consider the case where P_3 is within the segment P_1P_2. We have

at once
$$\frac{P_1P_3}{P_2P_3} = \frac{P_1S}{P_3R} = \frac{x_3 - x_1}{x_2 - x_3}$$

therefore
$$\frac{\lambda_1}{\lambda_2} = \frac{x_3 - x_1}{x_2 - x_3}$$

and
$$x_3 = \frac{\lambda_1 x_2 + \lambda_2 x_1}{\lambda_1 + \lambda_2}$$

PROBLEMS.

1. Show from the same figure that $y_3 = \frac{\lambda_1 y_2 + \lambda_2 y_1}{\lambda_1 + \lambda_2}$.

2. Construct the figure for the case when P_3 is without the segment P_1P_2, and show that the same formulae hold.

3. In the work of both paragraphs 21 and 22 the axes have been rectangular. Deduce the corresponding formulae when the axes are oblique.

4. Find the distance from (2, 3) to (5, —1); (4, 1); (—5, 1); (—1, —1); (0, 3).

5. What is the general formula for the distance of a point from the origin?

6. Express the co-ordinates of the middle point of a segment of a line in terms of the co-ordinates of its extremities.

7. Find the co-ordinates of the points which divide the segment of the line terminating at (1, 3) and (4, —2) into three equal parts and find the length of these parts.

8. Find the co-ordinates of the points which have the given distance ratios with respect to the following pairs of points. The shorter segment is in each case the one terminating at the point first named.

Points.		Ratios.
(1, 4)	(2, 3)	$\frac{2}{3}$
(3, 2)	(—1, 0)	$\frac{1}{2}$
(2, —1)	(—2, —4)	2
(5, 6)	(1, —3)	$-\frac{3}{4}$
(0, 0)	(—2, 5)	$-\frac{7}{2}$

23. ANGLE BETWEEN A LINE AND THE X AXIS. In analytic geometry the angle which a line makes with the X axis is always measured from the positive end of the X axis toward the positive end of the Y axis. Remembering this, the student should have no difficulty in showing that the tangent of the angle made with the X axis by the line through two points is given by the formula,

$$\tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$

where θ is the angle and (x_1, y_1), (x_2, y_2) are the points. He should also show that this formula holds for all possible positions of the line.

PROBLEM.

Find the angles which the lines of problem 8 of the last paragraph make with the X axis.

24. ILLUSTRATIONS OF THE DEVELOPMENT OF EQUATIONS OF LOCI. Now that we have developed our formulae for distance, distance ratio, and direction of a straight line we are ready to take up again the problem we were compelled to leave unfinished in the latter part of paragraph 20. We now are able to translate the phrase, "The distance from (x, y) to $(4, 3)$" by the expression $\sqrt{(x-4)^2+(y-3)^2}$, and the statement of the way in which the variable point traces the curve now translates into

$$\sqrt{(x-4)^2+(y-3)^2}=2$$

which is the equation of the curve, since it is the algebraic statement of the necessary and sufficient condition that the point (x, y) may lie on the circle.

It is usual to reduce such expressions however to the simplest possible form, and the equation of this circle would usually be written in the form

$$x^2 + y^2 - 4x - 2y - 11 = 0.*$$

Let us consider a second example of this sort. The line joining a variable point to the point (1, 2) makes with the X axis an angle whose tangent at any instant is equal to the abscissa of the variable point at the same instant. Find the locus of the variable point. "The variable point" translates into (x, y). "The abscissa of the variable point" translates into x. "The tangent of the angle made by the line with the X axis" translates (by paragraph 23) into $\frac{y-2}{x-1}$. Therefore the algebraic statement of the condition under which the curve is described is evidently $\frac{y-2}{x-1} = x$

i. e., $$y = x^2 - x + 2.$$

If the student desires to know the form of the curve it can easily be plotted. Later on he will learn to classify simple curves without plotting.

PROBLEMS.

1. Find the locus of all points equally distant from (1, 1) and (2, 4); from (1, 3) and (—1, 5).
2. Generalize problem 1 by taking (x_1, y_1) and (x_2, y_2) as the two fixed points, and show that the equation is always of the first degree.
3. Find the equation of the circle whose center is at (a, b) and whose radius is r. (Since proper choice of a, b, and r will make this any circle whatever, the corresponding equation is called the general equation of the circle.)
4. Generalize the second illustration of this paragraph by replacing the point (1, 2) by a point (x_1, y_1).
5. A point moves so that the square of its distance from (3, 2) plus the square of its distance from (1, 3) equals 27, find the equation of the locus. Does a comparison of the result with that of problem 3 give any hint as to the nature of the curve?
6. Show that if a point moves so that the sum of the squares of its distances from three fixed points is constant, the equation of

---

*See appendix C.

its path will always be of the second degree, will have no term in xy, and will have the same coefficient for the terms in x^2 and y^2. Can these statements be extended to a greater number of points?

7. Q_1 and Q_2 are two fixed points and P is a variable point. The movement of P is subject to the condition that the tangents of the angles which the two lines PQ_1 and PQ_2 make with the X axis shall be numerically equal but of opposite sign. Find the locus of P.

8. A moving point traces a straight line passing through the point $(1, -2)$ and making with the X axis an angle whose tangent is 2. Show that the equation of the line is

$$y = 2x - 4$$

9. A line has an intercept on the Y axis of 4 (i. e., passes through the point $(0, 4)$) and makes with the X axis an angle whose tangent is 3. Show that its equation is

$$y = 3x + 4.$$

10. Since any line may be defined by its intercept on the Y axis and the angle it makes with the X axis, generalize problem 9 and show that the equation of any straight line is of the first degree and of the form

$$y = mx + h.$$

What are m and h?

11. Show conversely that any equation of the first degree in x and y can be reduced to the form

$$y = mx + h$$

and therefore represents a straight line.

The work of this paragraph will be resumed after the student has acquired a greater amount of material on which to base problems.

CHAPTER VII.

TO DEDUCE THE EQUATION WHEN A FINITE NUMBER OF POINTS ON A LOCUS OF SOME KNOWN TYPE IS GIVEN.

25. LIMITATIONS OF THE PROBLEM. OUTLINE OF THE METHOD.

If we are given merely a number of points on a locus there is no way by which the equation may be deduced, but if in addition to a number of points we are given such additional information as will enable us to determine the general form of the equation, the problem is at once simplified. For if the form of the equation is known, each point that satisfies the equation gives us a relation connecting the coefficients; and the complete determination of the equation is therefore possible whenever a sufficient number of points have been given. For example, let it be known that a straight line passes through (4, 7) and (3, 5). Problem 10 of the last paragraph tells us that the equation must be of the first degree and therefore of the general form

$$Ax + By + C = 0.^*$$

Now if the two points lie on the locus their co-ordinates must satisfy the equation and we have

$$4\frac{A}{C} + 7\frac{B}{C} + 1 = 0$$

$$3\frac{A}{C} + 5\frac{B}{C} + 1 = 0$$

whence $$\frac{A}{C} = -2 \quad \text{and} \quad \frac{B}{C} = 1$$

and the equation of the line is

$$-2x + y + 1 = 0.$$

*This form contains apparently three arbitrary constants, but division by any one of them reduces the equation to a form which contains only two. Such an equation is said to contain two effective constants.

Again, problem 3 of the last paragraph shows us that the equation of the circle whose center is at (a, b) and whose radius is r is of the form

$$(x-a)^2+(y-b)^2=r^2.$$

This equation contains three effective constants, but it is of the second degree in these constants, and our subsequent work will therefore gain in simplicity if we reduce the equation to the form

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0,$$

or, putting $a^2+b^2-r^2=-c$,

$$x^2+y^2-2ax-2by-c=0.$$

If now the three points (1, 1), (3, 3), (4, 1) lie on the curve we have

$$\begin{aligned} 2a+2b+c&=2 \\ 6a+6b+c&=18 \\ 8a+2b+c&=17 \end{aligned}$$

whence the student may find a, b, c and so determine the equation of the circle.

26. NUMBER OF POINTS REQUIRED TO DETERMINE A CURVE. The fact that the co-ordinates of every point on a curve must satisfy the equation of the curve, taken in connection with the algebraic theorem that n non-homogeneous equations are necessary and sufficient to determine n unknown quantities, leads us at once to the important theorem:

The number of effective arbitrary constants in an equation is equal to the number of arbitrary points through which the corresponding curve may be made to pass.

PROBLEMS.

1. Find the equations of the straight lines through the following pairs of points:

(1, 3) (2, —1); (2, 4) (3, 0); (4, 3) (2, 4); (4, 3) (—4, —3).

In the last case the values of $\frac{A}{C}$ and $\frac{B}{C}$ are infinite. This of course means merely that C is zero. The difficulty may be avoided by dividing by A in place of C.

2. Find the equation of the circle through the points (1, 2), (2, 4), (1, 4).

3. A curve of the form $y^2=2px$ passes through the point (4, 2). Determine the value of p.

4. Show that the equation of any straight line through the origin is of the form $y = mx$ where m is an arbitrary constant. What is the geometric significance of m?

5. Find the equation of the circle through the three points $(-1, 2)$, $(-1, -3)$, $(0, 0)$.

6. Show that the equation of any curve which passes through the origin can have no constant term.

7. How many effective constants are there in each of the following equations?

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$
$$(ax + by + c)(dx + fy + g) = 0$$
$$a(bx + cy + d) = 0$$
$$(a + b)x + cy = 0$$
$$ax^2 + by - cx + 2 = 0$$

27. COMPLEX CONDITIONS EQUIVALENT TO TWO OR MORE SIMPLE ONES.

In place of giving points on the curve, some other condition may be stated which is equivalent to giving one or more points. For example, to give the center of a circle is equivalent to giving both a and b, and therefore is equivalent to giving two points on the curve. Again, to give the angle which a line makes with the X axis is to determine m, and therefore is equivalent to giving one point.

PROBLEMS.

1. Find the equation of the circle which passes through $(3, -1)$ and has its center at $(4, 2)$.

2. Find the equation of the circle which passes through $(3, -1)$ and $(1, 4)$ and has its center on the X axis.

3. Find the equation of the straight line through the point $(3, 4)$, making an angle with the X axis of 70°, 110°, 45°, 135°.

4. A given line makes with the X axis an angle whose tangent is m, and has an intercept a on the X axis. Show that its equation is

$$y = m(x - a).$$

5. Find the equation of a line through $(1, 5)$, making an angle of 45° with the X axis.

6. Find the equations of the lines through $(-2, 5)$ parallel to the lines of problem 3.

7. Find the equation of the line through the point $(-3, -4)$ parallel to the X axis, to the Y axis.

CHAPTER VIII.

TRANSFORMATION OF CO-ORDINATES.*

28. THE GENERAL PROBLEM. If the student turns back to paragraph 10, he will note that our present system of co-ordinates rests on certain assumptions which are equivalent to the arbitrary determination of the following: the origin, the direction of one axis, the angle between the axes, the scales of measurement. The exigencies of the discussion may at any time demand a change in any one of these, and the algebraic significance of such a change must therefore be investigated. As in the case of our work in one dimension, we desire to find formulae which will give us the old variables in terms of the new. We consider in turn the formulae corresponding to changes in each of the four assumptions mentioned above.

29. MOVEMENT OF THE AXES PARALLEL TO THEMSELVES. To change the first assumption without producing any change in any of the others it is sufficient to move the axes parallel to themselves. Let XOY be the original system and by such a movement secure a second system $X'O'Y'$, where the co-ordinates of O' referred to XOY are a and b. Let P be any point in the plane and let the co-ordinates of P be (x, y) in the first system and (x', y') in the second. Then for all positions of P we have

$$x = x' + a$$
$$y = y' + b$$

which are accordingly the formulae of transformation.

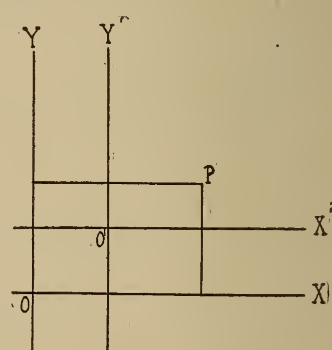

FIG. 11.

*The subdivision of our subject adopted in paragraph 12 seems to call at this point for a discussion of the last of the problems there stated, but the discussion is in many cases so facilitated by transforming the axes that it seems wise to introduce the present chapter at this point. The student who is not thoroughly familiar with the simpler theorems concerning the projection of plane contours should read Appendix D before undertaking the work of this chapter.

PROBLEMS.

1. Find the formulae corresponding to a change to a new set of axes parallel to the old, but with the new origin at the point $(1, -1)$, $(4, 0)$, $(0, 4)$, $(-2, -3)$, $(7, 1)$.

2. Find the formulae of transformation corresponding to the following movements of the axes:

The X axis 1 upward, Y axis 4 to the right;
The X axis 4 downward, Y axis unmoved;
The X axis unmoved, Y axis ½ to the left.

3. What movements of the axes correspond to the following substitutions?

$$\begin{cases} x = x' + 4 \\ y = y' - 2 \end{cases} \qquad \begin{cases} x = x' - 2 \\ y = y' \end{cases}$$

$$\begin{cases} x = x' + 6 \\ y = y' - 2 \end{cases} \qquad \begin{cases} x = x' \\ y = y' + 4 \end{cases}$$

4. In the figure the axes are rectangular. Will the same formulae hold in case the axes are oblique?

5. By a movement of the X axis reduce the equation

$$y = 3x + 4$$

to an equation in x' and y' which has no constant term. (Assume the axis to be moved a distance a, make the substitution, and then determine a by equating the constant term in the transformed equation to zero.)

6. Do the same thing by a movement of the Y axis.

7. Free the equation

$$x^2 + y^2 - 2x - 5y + 1 = 0$$

from the terms of the first degree in x and y by a movement of the origin, keeping the axes parallel to their original position.

8. Move the origin so that the equation

$$by^2 + 2fy + 2gx + c = 0$$

shall be transformed to an equation having no term of the first degree in y and no constant term. Is such a movement always possible?

9. What must be the co-ordinates of the new origin in order that the most general equation of the second degree

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

may reduce to an equation which has no terms of the first degree in the variables?* Is the operation possible when

$$h^2 = ab?$$

10. Are there any terms of an equation which cannot be removed by a transformation of the kind we have been considering?

30. ROTATION OF THE AXES. To change the second assumption without affecting any of the others it is sufficient to rotate the axes about the origin into a new position in which they make an angle θ with the old position.

Let XOY be the axes in the first position and $X'O'Y'$ the axes in the new position. Let P be any point having the co-ordinates (x, y) and (x', y') in the two systems. Then

$$Oa = x,\ Pa = y,$$
$$Oc = x',\ Pc = y'.$$

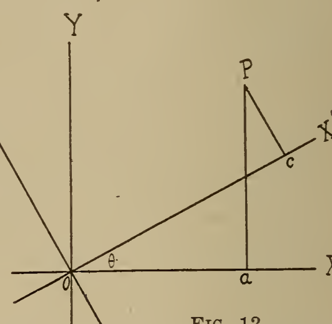

FIG. 12.

The two contours OaP and OcP have the same terminal points and their projections on any line are therefore equal. Hence we have, by projecting on OX,

$$Oa \cos 0 + aP \cos \frac{\pi}{2} = Oc \cos \theta + cP \cos \left(\theta + \frac{\pi}{2}\right).$$

Similarly by projecting the same contours on the line OY we have

$$Oa \cos \frac{\pi}{2} + aP \cos 0 = Oc \cos \left(\frac{\pi}{2} - \theta\right) + cP \cos \theta,$$

and these two equations at once reduce to

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta$$

which are the desired formulae of transformation.

*If an equation of the second degree has no terms of the first degree in the variables it is of the form

$$ax^2 + by^2 + 2hxy + c = 0.$$

If such an equation is satisfied by a point (x_1, y_1) it will also be satisfied by the point $(-x_1, -y_1)$. The locus is therefore symmetrical with respect to the origin, and our problem might be thus stated: To move the origin to the center of symmetry of the curve represented by the equation of the second degree.

If we desire the formulae which correspond to a change from $X'OY'$ to XOY we may solve these equations for x' and y', or we may project on the lines OX' and OY', or we may note that this new case differs from the original one only in the angle, which is now negative. Any one of these methods will give us

$$x' = x \cos\theta + y \sin\theta$$
$$y' = -x \sin\theta + y \cos\theta$$

PROBLEMS.

1. Write the formulae of transformation which correspond to the following rotations, putting in in each case the numerical values of cos θ and sin θ

$$60°, 30°, 45°, -30°, 150°, (\pi - 60°), \left(\frac{3}{2}\pi - 30°\right), \frac{\pi}{2}, \pi.$$

2. Show that the equation

$$x^2 - xy - 2 = 0$$

may be freed from the term in xy by a proper rotation of the axes. (Apply the proper formulae and the equation becomes

$$x'^2(\cos^2\theta + \cos\theta\sin\theta) + y'^2(\sin^2\theta - \cos\theta\sin\theta)$$
$$+ x'y'(\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta) - 2 = 0.$$

In order that there may be no term in $x'y'$ we must so choose θ that

$$\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta = 0$$

or

$$\cos 2\theta - \sin 2\theta = 0$$
$$\tan 2\theta = 1$$
$$\theta = 22° \, 30' = \frac{\pi}{8}.$$

3. Free the equation

$$x^2 - xy + 3y - 2x = 0$$

from the term in y by a rotation of the axes.

4. How many terms may be removed from an equation by a rotation of the axes? Are there any terms which are unaffected by such a rotation?

5. Through what angles may the axes be turned without introducing an xy term into the equation

$$x^2 - y^2 = r^2?$$

6. Show that the general equation of the second degree will reduce to an equation without an xy term if the axes are rotated through an angle θ such that $\tan 2\theta = \dfrac{2h}{(a-b)}$.

31. CHANGE OF THE ANGLE BETWEEN THE AXES. To change the third assumption without affecting any of the others it is sufficient to change the direction of either of the axes, but it will be better to develop the more general formulae corresponding to a change of the directions of both axes.

Let XOY be the first set of axes and $X'OY'$ the second. Let OX' make an angle α and OY' an angle β with OX, and let the angle between OX' and OY' be δ ($\delta = \beta - \alpha$). Let P be any point (x, y) in the plane. Then by considering the projections of OaP and OcP on OX and OY we have

$$x = x' \cos \alpha + y' \cos \beta$$
$$y = x' \sin \alpha + y' \sin \beta,$$

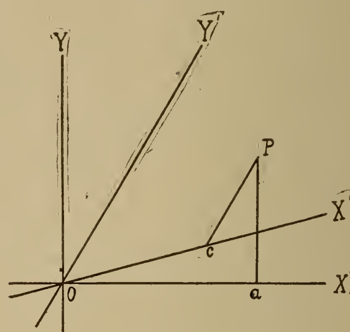

FIG. 13.

a set of formulae which are sufficient to transform from any set of rectangular axes to any set of oblique axes which have the same origin.

If we wish to transform from oblique to rectangular axes we have only to solve the formulae just derived for x' and y' and we have

$$x' = \frac{x \sin \beta - y \cos \beta}{\sin \delta}$$
$$y' = \frac{-x \sin \alpha + y \cos \alpha}{\sin \delta}$$

PROBLEMS.

1. Free the equation

$$2x - 3y + 1 = 0$$

from its term in y by a change in the direction of the Y axis.

2. What form does the equation

$$x^2 + y^2 = r^2$$

take when the X axis is left unchanged but the Y axis moved so that the angle between the axes is $30°$?

32. CHANGE OF THE SCALES OF MEASUREMENT. To change the last of our assumptions without affecting any of the others it is sufficient to write

$$x = Kx'$$
$$y = Ly'.$$

The effect is evidently to multiply the unit of measurement on the X axis by K and on the Y axis by L.

33. THE GENERAL TRANSFORMATION. The transformation from any system of axes to any other may now be accomplished by the proper combination of the formulae developed in the preceding paragraphs.

PROBLEMS.

1. Write the formulae of transformation which correspond to a movement of the origin to the point (4, 5) and a rotation of the axes through 60°.

2. Write the formulae which correspond to the transformation from rectangular axes to a new set of oblique axes, whose origin is at (—4, 3) and whose X and Y axes make angles of 30° and 95° with the original X axis.

3. The origin is moved to the point (—2, —4), and the axes are rotated through an angle of 10°. Write the formulae of transformation.

4. Show that none of the transformations so far discussed can change the degree of an equation. (It is sufficient to show that it cannot be raised. For if a change of the axes transforms an equation into one of lower degree, change the axes back to their original position and the equation will be restored to its original form, i. e., the degree of the transformed equation will be raised. Therefore when the student has shown that the degree of an equation cannot be raised he has shown that it cannot be lowered.)

34. TRANSFORMATIONS INTERPRETED AS CHANGES OF THE LOCI. Each of the above transformations has been interpreted as corresponding to a change in the system of co-ordinates. There is however another interpretation which is frequently adopted. Consider any equation as referred to a given system of reference. Apply any one of the above substitutions. The result will be a new equation, which may of course be referred to the original system of reference and

when so referred represents a new locus. The transformation may thus be regarded as a change in the locus instead of a change in the system of reference. For example, apply to the circle

$$(x-a)^2+(y-b)^2=r^2$$

the substitution

$$x=x'+a \qquad y=y'+b$$

and refer the resulting equation

$$x'^2+y'^2=r^2$$

to the original set of axes. From this point of view the transformation has evidently resulted in moving the center of the circle from the point (a, b) to the origin. The student will find it interesting to study all of the above transformations from this second point of view.

CHAPTER IX.

INTERSECTION OF LOCI.

35. THE SIGNIFICANCE OF IMAGINARIES. We have found that any equation connecting x and y represents a locus every point of which satisfies the equation. The following question, which was given a somewhat superficial treatment in paragraph 11, now demands more careful consideration. "Given two equations, is it possible to find a point or points which lie on both loci and therefore satisfy both equations?" Looked at from the geometric side the question is, "Do the loci intersect?" and the answer is, "They may or may not according to their relative positions." Looked at from the algebraic side the question is, "Can two equations in two variables be simultaneously satisfied?" and the answer is, "Yes, without exception." The cause of this apparent discrepancy lies in the nature of our fundamental assumptions, which were so made as to establish a one to one correspondence between the points of the plane and pairs of real values of x and y, while the theorem that two equations in two variables can always be simultaneously satisfied holds true only when pairs of imaginary values are included.

To make the matter a little clearer consider the equation

$$x - y = 0.$$

Pairs of values that will satisfy this equation are of two kinds; real values such as $x = a$, $y = a$, or complex values such as $x = a + ib$, $y = a + ib$, including pure imaginaries $x = ib$, $y = ib$. The first kind alone corresponds to points in the plane and includes all the points on the line through the origin bisecting the angles in the first and third quadrants. It is evident that there is not a complete correspondence between this line and the equation $x - y = 0$, since the equation has a much more general significance than the line; and it is also evident that this lack of complete correspondence is due to the nature of our fundamental assumptions, which give us no geometric representation for pairs of imaginary values.

It is however frequently desirable to be able to state algebraic theorems in geometric language, and so mathematicians are accustomed to speak of these pairs of complex values of x and y as representing imaginary points. From this point of view the curve corresponding to any equation $f(x, y) = 0$ is considered to consist of: (a) an infinity of real points which constitute the visible curve; (b) an infinity of imaginary points just as intimately associated with the equation, but having no representation in the diagram. (See appendix E.)

Any point, real or imaginary, which belongs to each of two curves is called an intersection, real or imaginary, of the curves. From the algebraic theorem that two equations of degree m and n in two variables can always be satisfied by mn pairs of values of the variables it follows that two curves of degree m and n intersect in mn real or imaginary points, some of which may in special cases coincide with each other.

36. POSSIBILITY OF ERROR.

So long as both the equations are of the first degree no ambiguity in the results is possible. For example, the two equations

$$2x - y - 3 = 0$$

and

$$4x - y - 7 = 0$$

yield on solution the two equations

$$x = 2 \qquad y = 1$$

showing that the intersection is on the line parallel to the Y axis at a distance 2 to the right, and on the line parallel to the X axis at a distance 1 above.

But consider the intersections of the circle of Fig. 14, whose equation is

$$2x^2 + 2y^2 = 3$$

and the curve $A'B'C'D'$ whose equation is

$$x^2 + xy + y^2 = 2.$$

Solving these two equations for x we have

$$x = \pm\sqrt{\frac{6 \pm \sqrt{20}}{8}}$$

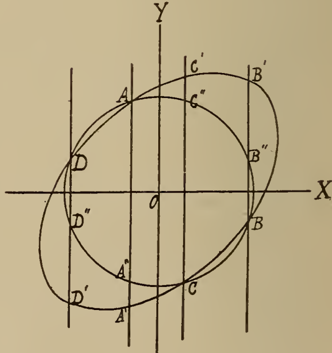

FIG. 14.

showing that the points of intersection are on one of the four lines AA', BB', CC', DD'. If now

$$-\sqrt{\frac{6-\sqrt{20}}{8}},$$

for example, be substituted for x in the equation of the circle we obtain, on solving the resulting equation in y, the y co-ordinate of the intersection A, but we also obtain the y co-ordinate of the point A'' in which we have no interest; while if the same value of x is substituted in the equation of the other curve, we obtain the y co-ordinates of the points A and A'. In such cases as this the student must determine by substitution in the equations which of the points obtained correspond to the actual intersections of the curves.

PROBLEMS.

Find the intersections of the following pairs of curves:

1. $2x + 3y + 1 = 0$
   $4x - y + 2 = 0$
2. $x - y = 0$
   $x + y = 0$
3. $x - 2y + 1 = 0$
   $-4x + 2y - 7 = 0$
4. $x - iy + 2 = 0$
   $x + iy + 4 = 0$
5. $x - 2iy + 3 = 0$
   $x + 2iy + 3 = 0$
6. $x^2 + y^2 = 4$
   $x + 2y = 0$
7. $x^2 + y^2 + 4 = 0$
   $x + 2y + 1 = 0$
8. $x^2 + y^2 - 2 = 0$
   $x + y - 4 = 0$
9. $4x^2 + 3y^2 - 12 = 0$
   $3x^2 - 4y^2 - 12 = 0$
10. $4x^2 + 9y^2 = 36$
    $9x^2 + 4y^2 = 36$

11. Find the intersections of

$$y = a$$

with

$$x^2 + y^2 = b^2$$

and state in both algebraic and geometric terms what happens as a, at first less than b, gradually increases till it is greater than b.

12. Find the intersection of

$$y = m_1x + b_1$$

and

$$y = m_2x + b_2$$

and discuss the case when $m_1 = m_2$.

Find the intersections of the following curves with the axes:

13. $y = mx + b$
14. $\dfrac{x}{a} + \dfrac{y}{b} = 1$
15. $Ax + By + C = 0$
16. $x^2 - 2ay + 1 = 0$

CHAPTER X.

THE EQUATION OF THE FIRST DEGREE AND THE STRAIGHT LINE.*

37. STANDARD FORMS OF THE EQUATION OF THE STRAIGHT LINE. Our investigation of the geometric properties and relations of loci by means of their equations will be facilitated by adopting some mode of classification of equations. That by degrees is probably the most natural and for our present purpose the most convenient.

The student has already deduced for himself a number of important results concerning the equation of the first degree and its corresponding locus which are here summarized for convenience of reference.

I. Every equation of the first degree represents a straight line. (Problem 11. Paragraph 24.)

II. Conversely, every straight line is represented by an equation of the first degree. (Problem 10. Paragraph 24.)

III. If the equation be of the form

$$y = mx + h$$

m is the tangent of the angle made by the line with the X axis and h is the intercept on the Y axis. (Problem 10. Paragraph 24.)

IV. If the equation be of the form

$$y = m(x - a),$$

m has the same meaning as before, but a is now the intercept of the line on the X axis. (Problem 4. Paragraph 27.)

---

*We begin at this point the study of the last of the five problems mentioned in paragraph 12 and shall continue it for several chapters. The student must remember, however, that any such division of a subject as is attempted in stating these five heads is of necessity somewhat arbitrary, and he must therefore not be surprised to find under this last head problems and theorems that might with entire propriety be stated under some other heading. In particular he will find an entire chapter (Chapter XII) devoted to loci problems which have been deferred to this later position because the student had not at an earlier date sufficient material on which to base them.

V. If the equation be of the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

a and b are the intercepts on the X and Y axes. (Problem 14. Paragraph 36.)

The tangent of the angle made by the line with the X axis is frequently called the slope of the line, and the forms in III and IV are consequently called the slope equations of the straight line, while the form in V is called the intercept equation of the straight line. $Ax + By + C = 0$ is called the general equation of the straight line.

PROBLEMS.

1. Find the slopes and intercepts of the following lines:

$$2x + 3y + 1 = 0 \qquad y = 4x + 2$$
$$x - 2y - 4 = 0 \qquad x = 3y + \frac{1}{3}$$
$$3x + \frac{2}{3}y - \frac{1}{2} = 4 \qquad y = 3$$
$$x - 2y = 0 \qquad x = a$$

2. Compare the general equation with the slope and intercept equations and deduce the following results for the general equation

$$\text{Slope} = -\frac{A}{B}$$

$$\text{Intercept on the } X \text{ axis} = -\frac{C}{A}$$

$$\text{Intercept on the } Y \text{ axis} = -\frac{C}{B}$$

3. A knowledge of the intercepts makes the plotting of the line an easy matter, excepting in one special case. (What is this exception?) Plot the lines of problem 1.

4. Since the slope of a line depends on the ratio $\frac{A}{B}$ what conclusion may be drawn as to the equations of parallel lines?

5. Let the lines AB and CD have the equations

$$y = m_1x + h_1$$

and

$$y = m_2x + h_2$$

respectively. Show that

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

where θ is the angle between the lines.

6. Find the angles between the following pairs of lines:

$$x + 2y - 1 = 0 \qquad 2x - 2y + 1 = 0$$
$$x - 3y + 4 = 0 \qquad x = 3y + 2$$

$$4x + y - 2 = 0 \qquad 2x - 2y + 1 = 0$$
$$y = 3x - 4 \qquad x - 6y + 4 = 0$$

$$x + y - 2 = 0 \qquad x - 2y + 3 = 0$$
$$4x + 3 = -4y \qquad y + 2x + 5 = 0$$

7. Show that the angle between the two lines

$$A_1x + B_1y + C_1 = 0$$

and

$$A_2x + B_2y + C_2 = 0$$

is given by

$$\tan\theta = \frac{B_1 A_2 - B_2 A_1}{A_1 A_2 + B_1 B_2}$$

8. From the results of problem 5 determine what conditions must be satisfied by the coefficients of

$$y = m_1x + h_1$$

and

$$y = m_2x + h_2$$

in order that the lines may be parallel. Perpendicular.

9. Determine the corresponding conditions for

$$A_1x + B_1y + C_1 = 0$$

and

$$A_2x + B_2y + C_2 = 0$$

10. Remembering that

$$\sin\theta = \frac{\tan\theta}{\pm\sqrt{1 + \tan^2\theta}} \quad \text{and} \quad \cos\theta = \frac{1}{\pm\sqrt{1 + \tan^2\theta}}$$

deduce formulae for the sine and cosine of the angle which a line makes with the X axis in terms of m and h. In terms of A and B.

11. A line is subject to the condition that it must be parallel to the line

$$2x - 3y + 1 = 0.$$

To what extent are its coefficients determined and to what extent are they still arbitrary?

12. Find the equations of lines through the point (2, 3) parallel and perpendicular to each of the following lines:

$$x - 2y + 3 = 0 \qquad x - 2y + 3 = 0$$
$$x = 3y + 4 \qquad x = 0$$
$$-y + 2x = 9 \qquad x - y = 0$$
$$x = 4y \qquad ax - cy + f = 0$$

13. Are there any cases of parallelism or perpendicularity among the following lines?

$$x-2y+3=0 \qquad x+4y+12=0$$
$$y=4x+5 \qquad 6y-3x+1=0$$
$$7x-2y+3=0 \qquad y-4x+3=0$$

14. A line is subject to the condition of passing through the point (1, 3). To what extent are its coefficients still undetermined?

15. Find by the method of paragraph 25 the general equation of a straight line through the point (x_1, y_1).

38. THE EQUATION OF THE STRAIGHT LINE THROUGH TWO GIVEN POINTS. If we attempt to find the equation of the straight line through the two points (x_1, y_1) and (x_2, y_2), the method of paragraph 25 is of course perfectly rigorous, but it has the disadvantage of demanding an amount of algebraic work which is somewhat wearisome, and in certain similar but somewhat more complex investigations becomes well nigh prohibitive. It is sometimes possible in such cases to avoid much of this algebraic work and infer the form of the equation desired from principles already established. Consider for example problem 15 of the last paragraph. The equation desired must be of the first degree, must be satisfied by the point (x_1, y_1), and must be sufficiently general in its form so that we may be able to satisfy it by the co-ordinates of any other one point in the plane. The equation must of course not be identically zero. Any equation, no matter how secured, that possesses these properties is the equation desired. Now it is easy enough to manufacture an equation which possesses them.

$$y-y_1=m(x-x_1)$$

in fact possesses all these properties and will hereafter be used as the equation of any line through (x_1, y_1). (What is the geometric significance of m?)

The problem stated at the heading of this paragraph may be treated in the same way. Here the equation must be of the first degree, must be satisfied by (x_1, y_1) and (x_2, y_2), must contain no arbitrary constant (Why?), and must not vanish identically.

$$(y-y_1)(x_2-x_1)=(x-x_1)(y_2-y_1)$$
$$(y-y_2)(x_1-x_2)=(x-x_2)(y_1-y_2)$$

meet these conditions, and either of them may therefore be taken as the desired equation. The same result may be obtained by substituting in

$$y - y_1 = m(x - x_1)$$

the value of m deduced in paragraph 23. These equations are more frequently written in one or the other of the two forms

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Still another form of this equation is frequently met in mathematical literature. It may be deduced as follows: If the point (x, y) is to trace a straight line it must satisfy an equation of the form

$$ax + by + c = 0,$$

and if the line is to pass through the points (x_1, y_1) and (x_2, y_2) we must have

$$ax_1 + by_1 + c = 0$$

and

$$ax_2 + by_2 + c = 0$$

The co-existence of these three equations is the necessary and sufficient condition that the point (x, y) may trace a straight line through the two points (x_1, y_1) and (x_2, y_2). But the necessary and sufficient condition for the co-existence of these three equations is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which is therefore the equation of the line.

39. THE STRAIGHT LINE GIVEN BY TWO EQUATIONS IN THREE VARIABLES. Consider a line through (x_1, y_1) making the angles α and β with the axes, and let (x, y) be a variable point on the line. The necessary and sufficient conditions which must be satisfied in order that the variable point may be on the line are

$$x - x_1 = r \cos \alpha$$

and

$$y - y_1 = r \cos \beta$$

where r is the variable distance from (x, y) to (x_1, y_1). These two equations may be written in the form

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = r,$$

or

$$x = x_1 + r\cos\alpha$$
$$y = y_1 + r\cos\beta$$

These two equations are used in cases where the student is interested in the distance from the tracing point to some fixed point on the line. r can be readily eliminated between the two equations and any one of the forms of the equation previously discussed at once deduced. Hereafter we shall refer to this form as the parametric form of the equations of the straight line.

40. DISTANCE FROM A POINT TO A LINE. The problem of finding the distance from a given point (x_1, y_1) to a given line

$$Ax + By + C = 0$$

might now be solved by writing the equation of a line through (x_1, y_1) perpendicular to

$$Ax + By + C = 0,$$

finding the point of intersection of these two lines, and determining the distance from this point of intersection to the point (x_1, y_1). But a geometric solution is simpler and leads to an important algebraic result.

Let P be any point (x_1, y_1), and AB any line. Let p' denote the perpendicular distance from the point P to the line AB, and p the perpendicular distance from the origin to the same line. Let α and β denote the angles which the perpendicular from the origin

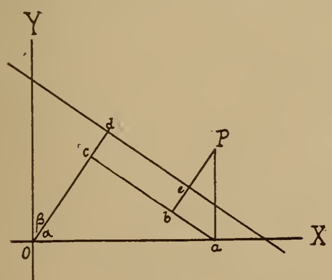

FIG. 15.

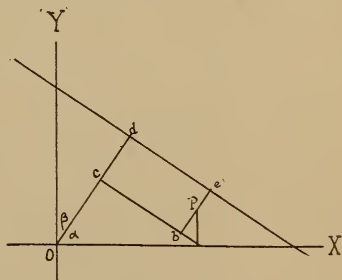

FIG. 16.

on the line AB makes with the axes. Draw Pa perpendicular to OX, Pb and Od perpendicular to AB, and ac parallel to AB.

Then $Oc + bP - Od = Pc$

or $Oc + bP - Od = -Pc$

i. e., $x_1 \cos\alpha + y_1 \sin\alpha - p = p'$ or $-p'$

according as the point P is on the opposite or same side of the line AB as the origin. This formula gives the distance from the point to the line in terms of the co-ordinates of the point and the constants p and α which determine the position of the line.

41. NORMAL FORM OF THE EQUATION OF A STRAIGHT LINE. The necessary and sufficient condition that any point (x, y) be on the line, (i. e. the equation of the line) is that p' for that point shall be zero.

$$x \cos\alpha + y \sin\alpha - p = 0$$

is therefore another form of the equation of the straight line. It is called the normal form of the equation.

In order to reduce the general equation to the normal form we multiply it by an undetermined constant m. Then if

$$mAx + mBy + mC = 0$$

is identical with

$$x \cos\alpha + y \sin\alpha - p = 0$$

we must have

$$mA = \cos\alpha$$
$$mB = \sin\alpha$$
$$mC = -p$$

Therefore $$m^2A^2 + m^2B^2 = 1$$

and $$m = \frac{1}{\sqrt{A^2 + B^2}}.$$

Substituting this value we have for the normal form of the equation

$$\frac{Ax}{\sqrt{A^2 + B^2}} + \frac{By}{\sqrt{A^2 + B^2}} + \frac{C}{\sqrt{A^2 + B^2}} = 0.$$

The results of the last two articles may evidently be stated in the following condensed form. The distance of any point from a given straight line is the value obtained by substituting the co-ordinates of the point in the left hand member of the normal form of the equation of the line.

PROBLEMS.

1. Write the equations of each of the following lines in each of the forms already developed.

Through the point	*Angle with X axis.*
(1, 3)	$\frac{\pi}{3}$
(1, 4)	3 radians
(—2, —4)	65 degrees
(a, b)	c degrees

Through the Points

(1, 4)	and	(2, —3)
(1, —6)		(α, β)
$(4, c)$		(0, 2)
(4, 5)		(4, —3)
(4, 2)		$(-a, 2)$

Through the origin parallel to the first of the list.

Through (1, 1) perpendicular to the first of the list.

2. Find the distance from the origin to each of the lines in the first group above.

3. Find the distance from the point (1, 1) to each of the lines of the first group above. In which cases are the origin and (1, 1) on opposite sides of the line?

4. Find the area of the triangle whose vertices are (0, 0), (1, 2), (3, 1).

5. Find the area of the triangle whose vertices are (1, —1), (3, 2), (—3, —2).

6. If A is the area of a triangle and (x_1, y_1), (x_2, y_2), (x_3, y_3) are its vertices, show that

$$2A = y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)$$

7. In determinant notation this becomes

$$2A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Deduce this form directly by use of the determinant form of the equation of a straight line.

8. To say that three points lie on a straight line is evidently equivalent to saying that the area of the triangle formed by them is zero. Show that the necessary and sufficient condition for collinearity of three points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

9. Deduce the same conclusion from the fact that the necessary and sufficient condition that the three points may lie on the line

$$Ax + By + C = 0$$

is that the co-ordinates of each point shall satisfy the equation of the line.

PROBLEMS.

42. INTERSECTIONS OF LINES.

Find the intersections of the following lines:

1. $3x - 2y + 2 = 0$ and $x + y = 2$
2. $x - 3y + 1 = 0$ and $x - 4y - 1 = 0$
3. $x = 2y - 7$ and $y = 7x - 2$

4. Find the vertices of the triangle formed by the lines

$$y = 4x$$
$$2x - 3y = 4$$
$$5x - 4y + 2 = 0$$

5. Show that the necessary and sufficient condition which must be satisfied in order that the three lines

$$A_1x + B_1y + C_1 = 0$$
$$A_2x + B_2y + C_2 = 0$$
$$A_3x + B_3y + C_3 = 0$$

may meet in one point (i. e., be concurrent) is

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$$

In general any two lines intersect in one point. The only case in which an apparent exception is to be noted is when the lines are parallel. It is possible to include this apparent exception, however, under the general case. If the student will refer to his discussion of Problem 12, Paragraph 36, and take into consideration the geometric significance of m_1 and m_2 he will see that the result of that discussion may be stated thus. As two lines tend to parallelism one or both of the co-ordinates of the point of intersection must increase beyond all limit, i. e., the point of intersection removes indefinitely from the origin. Mathematicians are accustomed to use in place of this statement the abbreviated phrase, "Parallel lines intersect at infinity" and in this way are able to include the special case of parallelism in the general statement that any two lines intersect in a single point.

43. **FAMILIES OF LINES.** Let the straight lines represented by the two equations

$$A_1x + B_1y + C_1 = 0$$

and

$$A_2x + B_2y + C_2 = 0$$

intersect in the point (x_1, y_1). Then if μ and ν are any two constants,

$$\mu(A_1x + B_1y + C_1) + \nu(A_2x + B_2y + C_2) = 0$$

is the equation of a straight line (Why?) passing through the point (x_1, y_1), since the substitution of x_1 and y_1 for x and y causes each of the parentheses to vanish and therefore satisfies the equation. If we put

$$A_1x + B_1y + C_1 \equiv S_1$$
$$A_2x + B_2y + C_2 \equiv S_2$$
$$\frac{\nu}{\mu} = \lambda$$

we may write the third line in the abridged form

$$S_1 + \lambda S_2 = 0.$$

In general if $R_1 = 0$ and $R_2 = 0$ are the equations of any two loci,

$$R_1 + \lambda R_2 = 0$$

is the equation of a third locus passing through all the intersections of the first and second.

The equation

$$S_1 + \lambda S_2 = 0$$

contains one arbitrary constant as it properly should, since the line represented by it has been subjected to the single condition of passing through one point and therefore has one degree of freedom remaining. This constant may be determined so that the line passes through any other point in the plane, and therefore it is evident that the equation by proper choice of λ may be made to represent any line in the plane through the intersection of $S_1 = 0$ and $S_2 = 0$. The aggregate of all such lines is spoken of as the family of lines through the intersection of $S_1 = 0$ and $S_2 = 0$ and the equation

$$S_1 + \lambda S_2 = 0$$

is called the equation of the family. The arbitrary constant is

called a parameter, a name given to any constant entering into an equation and taking in the course of the discussion a succession of arbitrary values. Thus

$$y = mx$$
$$y = a$$
$$y - y_1 = n(x - x_1)$$

are respectively the equations of the families of lines which pass through the origin, are parallel to the X axis, and pass through the point (x_1, y_1). The respective parameters are m, a, and n.

If it is desired to determine a particular member of a family an additional condition must be given. When the equation of the family is subjected to this condition a value or set of values of the parameter will be determined which will determine the member or members of the family satisfying the condition. For example, suppose we wish to determine that member of the family

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0$$

which passes through the point (3, 2). We have at once

$$3A_1 + 2B_1 + C_1 + \lambda(3A_2 + 2B_2 + C_2) = 0$$

whence the value of λ is at once determined. Substituting this value of λ in the equation of the family we have the particular member of the family which we desire.

PROBLEMS.

1. Write the equation of the family of lines through the intersection of

$$2x - 3y + 1 = 0$$

and

$$5x - y + 2 = 0$$

and find that member of the family which

(a) passes through the point (1, 5)
(b) is perpendicular to $4x - 2y = 0$
(c) has an intercept of 4 on the X axis
(d) makes an angle of 30 degrees with the X axis.
(e) is parallel to the bisector of the 1st and 3rd quadrants at the origin.

2. Write the family of lines through the point (1, 2).
3. Write the family of lines parallel to the line

$$3x + 2y + 4 = 0.$$

44. THE LINE AT INFINITY. If the equation of a line be written in the intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

and the line be removed farther and farther from the origin, a

and b indefinitely increase and the coefficients of x and y tend to zero. In other words, as a line removes indefinitely from the origin its equation tends to the form

$$0x + 0y = 1$$

or, since the equation may be multiplied by any constant whatever, to the form

$$0x + 0y = C.$$

Moreover this form does not depend on the original position of the line or on the manner in which it is removed. In other words the equations of all lines tend to a single form as the lines are indefinitely removed from the origin, a statement which is equivalent to the assertion that all points at infinity satisfy the single first degree equation

$$0x + 0y = C.$$

But since in analytic geometry we deal with loci only through their equations, mathematicians are accustomed to express all this by the abbreviated phrase, "All points at infinity lie on a single straight line whose equation is

$$0x + 0y = C."$$

For any point in the finite part of the plane the left hand member of the above equation is of course zero, and since we are in almost all of our investigations concerned only with the finite part of the plane, it is customary in all such investigations to abbreviate still more the above form and write the equation of the line at infinity as

$$C = 0.$$

45. PARALLEL LINES. The subject of parallel lines affords an interesting application of this idea of the line at infinity. Let the line GH be drawn through the intersection of AB and CD. Then if the equations of AB and CD are respectively

$$S_1 = 0 \text{ and } S_2 = 0$$

the equation of GH is

$$S_1 + \lambda S_2 = 0.$$

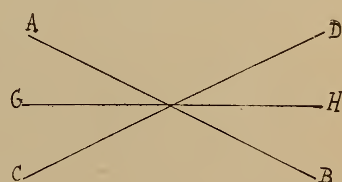

FIG. 17.

Let CD be removed indefinitely, its equation will tend to $C = 0$.

GH will tend to parallelism with AB, and its equation will tend to the form

$$S_1 + \lambda C = 0,$$

which gives our former theorem that the equation of a line parallel to a given line differs from the equation of the given line only in the constant term. A line parallel to a given line is, from this new point of view, a line through the intersection of the given line with the line at infinity.

CHAPTER XI.

THE CIRCLE, A SPECIAL CASE OF THE EQUATION OF THE SECOND DEGREE.

46. THE GENERAL EQUATION OF THE CIRCLE. The most general form of the equation of the second degree is

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

and the corresponding curves, from certain relations which they bear to the cone, are called conic sections or simply conics.

Before taking up the study of the general conic we shall consider the special case of the circle. Its equation has been already found (Problem 3, Paragraph 24), in the form

$$(x - a)^2 + (y - b)^2 = r^2$$

where (a, b) is the center and r the radius. Multiplying, transposing, and replacing the constant term by c, we reduce this equation to the form

$$x^2 + y^2 - 2ax - 2by + c = 0$$

or multiplying by an arbitrary constant

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0.$$

We therefore state the general theorem that every circle is represented by a second degree equation, without a term in xy, and with the same coefficient for the terms in x^2 and y^2. The student may deduce the converse theorem that every second degree equation without a term in xy and with the same coefficients for the terms in x^2 and y^2 represents a circle by showing that every such equation may be reduced to the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

PROBLEMS.

1. Compare the two forms

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

and

$$(x - a)^2 + (y - b)^2 = r^2$$

and hence show that the center is the point $\left(-\frac{G}{A}, -\frac{F}{A}\right)$ and the radius is $\frac{1}{A}\sqrt{G^2 + F^2 - AC}$.

2. Find the centers and radii of the following circles:

$$x^2 + y^2 - 4x + 5y - 12 = 0$$
$$3x^2 + 3y^2 - 2y = 0$$
$$x^2 + y^2 - 4x = 0$$
$$4x^2 + 4y^2 - 12x + 2y + 4 = 0$$
$$3x^2 + 3y^2 - 4x + 2 = 0$$
$$x^2 + y^2 + px + qy - s = 0$$

3. Find the equations of the circles with the following centers and radii.

Centers	*Radii*
(3, 4)	2
(4, 1)	$\frac{1}{2}$
(3, 0)	5
(—2, —1)	k
(0, —4)	$\sin\frac{\pi}{6}$

4. Write equations which by proper choice of the constants involved will represent any circle of radius 2 whose center is on the X axis, on the line $y = a$, on the line $y = 3x$, on the line $y = 3x + 2$, on the circle $x^2 + y^2 = 4$.

5. Write the equation of the circle centered on the curve $y^2 = 3x$ and passing through the points (2, 3) and (4, 4).

6. Apply to the circle

$$x^2 + y^2 = r$$

a transformation of co-ordinates which will make both the new axes tangent to the circle. What is the new form of the equation?

7. What is the form of the equation when the new axes are a diameter and the tangent at its extremity?

8. Write the equation of a circle centered at (3, 1) and tangent to the line

$$3x - 2y + 4 = 0$$

(See paragraph 41.)

9. Show by a geometric construction that the condition

$$(x - a)^2 + (y - b)^2 = r^2$$

is satisfied by the coordinates of all points on the circle of radius r centered at (a, b) and by no others.

47. INTERSECTIONS OF CIRCLES. We might find the intersections of two circles by a direct solution of the two equations for x and y, but this would introduce an unpleasant amount of algebraic work. Subtracting

$$x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$$

from

$$x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0$$

we have

$$2x(a_2 - a_1) + 2y(b_2 - b_1) + c_2 - c_1 = 0,$$

a new locus passing through the intersections of the two circles. (See Paragraph 43.) This locus is a straight line and must therefore be the common chord of the two circles. Our problem is now reduced to the simpler one, already solved, of finding the intersections of this line and either of the circles.*

When one of the circles lies wholly within or without the other the intersections are of course imaginary. It will probably surprise the student to find that the common chord is, however, always real. The explanation of the fact will be evident if he will find the equation of the line determined by a pair of conjugate imaginary points, e. g. $(a+ib, c+id)$ and $(a-ib, c-id)$.

PROBLEMS.

Find the intersections of the following pairs of circles:

1. $x^2+y^2+2x-3y+2=0$
   $(x-3)^2+(y-4)^2=16$
2. $(x-1)^2+(y-2)^2=4$
   $x^2+y^2=4$
3. $(x-1)^2+(y-2)^2=4$
   $(x+3)^2+(y+5)^2=1$
4. $(x-6)^2+(y-4)^2=4$
   $(x-2)^2+(y-1)^2=9$

48. TANGENTS AND NORMALS. While the definition sometimes given in elementary geometry of a tangent line to a circle as a perpendicular to a radius at its extremity is entirely correct and might be used as a basis for our discussion of the tangent, it is not a definition which admits of extension to other curves which we shall study. The tangent line to any curve may evidently be regarded as the limiting position of a secant line as two of the points of intersection of the curve and the secant tend to coincidence. Let a secant line meet a curve in two points P and Q, and let the point P tend to coincide with Q. The limiting

---

*Two circles have of course four points of intersection (paragraph 35), but two of them are always imaginary points at infinity. The subtraction above gave terms $0x^2$ and $0y^2$. To drop these as we did was to assume that x and y were to remain finite, and the resulting equation therefore is satisfied by the finite intersection of the circles, but not by the infinite ones.

position of the secant line as P tends to Q is the tangent at Q. This is sometimes expressed by saying that the tangent to a curve meets it in two coincident points.

The normal to a curve at any point is the perpendicular to the tangent at that point.

Among the questions that arise concerning tangents are two that decidedly outrank the others in importance.

1. Given the equations of a line and a circle, how shall we determine whether the line is tangent to the circle, or in other words, what is the condition which the coefficients of the line must satisfy in order that it may be a tangent line to the circle?

2. Given the co-ordinates of a point and the equation of a circle, what is the equation of a line through the point tangent to the circle?

49. CONDITION OF TANGENCY. Any satisfactory definition of the tangent will of course lead to the condition of tangency if properly considered. For example, the fact that the tangent is perpendicular to the radius at its extremity is equivalent to the statement that the perpendicular distance from the point (a, b) to the line

$$y = mx + h$$

must equal r if the line is tangent to the circle of radius r centered at (a, b). Applying this test we have as the condition of tangency

$$\frac{ma - b + h}{\sqrt{1 + m^2}} = r$$

or

$$h = \pm r\sqrt{1 + m^2} - ma + b$$

This method of deriving the condition of tangency is unfortunately applicable only to the circle, since the definition of tangency on which it is based does not hold for other curves. The definition of the tangent as a line meeting the curve in two coincident points holds however for all curves. In deducing the condition of tangency from this definition we find the intersections of the line and the curve. The co-ordinates of these intersections are given by ordinary algebraic equations in one variable and the necessary and sufficient condition for tangency is that these equations shall have equal roots. Thus

$$y = mx + h$$

meets

$$x^2 + y^2 = r^2$$

in two points whose x co-ordinates are given by the quadratic equation $$x^2+m^2x^2+2mxh+h^2-r^2=0.$$
The necessary and sufficient condition for coincidence of the points of intersection, and therefore for tangency of the line, is that this quadratic in x be a perfect square, i. e., that
$$m^2h^2-(h^2-r^2)(1+m^2)=0$$
or $$h=\pm r\sqrt{1+m^2}$$
as before.

50.
EQUATION OF THE TANGENT THROUGH A GIVEN POINT.

Given a point (x_1, y_1) and a circle
$$x^2+y^2=r^2,$$
to determine the equation of the tangent to the circle through the given point we first write the general equation of a line through the point
$$y-y_1=m(x-x_1)$$
or $$y=mx+y_1-mx_1.$$
If this line is tangent we must have from the last paragraph
$$y_1-mx_1=\pm r\sqrt{1+m^2}$$
i. e., $$m=\frac{x_1y_1\pm r\sqrt{x_1^2+y_1^2-r^2}}{x_1^2-r^2}.$$
Inserting these two values of m in the equation of the line we have the equations of the two tangents *from* (x_1, y_1) to the circle. If however (x_1, y_1) is on the circle, we have
$$x_1^2+y_1^2-r^2=0,$$
and therefore $$m=-\frac{x_1}{y_1}$$
Substituting this value of m in the equation of the line we have for the equation of the tangent *at* a point (x_1, y_1) on the circle
$$x^2+y^2=r^2$$
the form $$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$
i. e., $$yy_1+xx_1=r^2$$

This method of finding the equation of the tangent is theoretically general, but in the case of more complex equations we encounter serious algebraic difficulties. For the development of the equation of the tangent at a point on a curve a second method, based upon the definition of the tangent as the limiting position

of the secant, is worth our investigation. Consider then a point P, (x_1, y_1), on the circle, and give to x_1 and y_1 such increments Δx_1 and Δy_1, that the new point Q, $(x_1+\Delta x_1, y_1+\Delta y_1)$, so obtained shall also be on the circle. Then the line

$$y-y_1=\frac{\Delta y_1}{\Delta x_1}(x-x_1)$$

is the secant line through the two points (x_1, y_1) and $(x_1+\Delta x_1, y_1+\Delta y_1)$. If we can determine the limiting value of the ratio $\frac{\Delta y_1}{\Delta x_1}$ as Δx_1 and consequently Δy_1 tend to zero we shall have the slope and therefore the equation of the tangent at (x_1, y_1). Since both points are on the circle we have

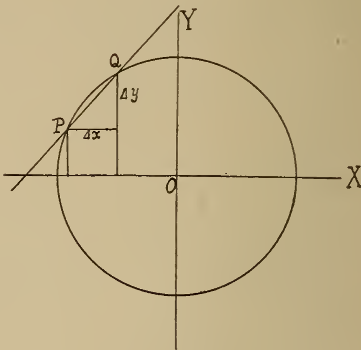

FIG. 18.

$$x_1^2+y_1^2=r^2$$

and $$x_1^2+2x_1\,\Delta x_1+\overline{\Delta x_1}^2+y_1^2+2y_1\,\Delta y_1+\overline{\Delta y_1}^2=r^2$$

Subtracting the first of these equations from the second we have

$$2x_1\Delta x_1+\overline{\Delta x_1}^2+2y_1\Delta y_1+\overline{\Delta y_1}^2=0$$

whence $$\frac{\Delta y_1}{\Delta x_1}=-\frac{2x_1+\Delta x_1}{2y_1+\Delta y_1}.$$

But the limit of this fraction as Δx_1 tends to zero is $-\frac{x_1}{y_1}$ which is therefore the limiting value of the slope of the secant line through the two points as the second point tends to coincidence with the first, i. e., the slope of the tangent at (x_1, y_1). The equation of the tangent is therefore

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

reducing as before to

$$yy_1+xx_1=r^2$$

This method is that employed in the differential calculus, and by the aid of the processes elaborated in the discussion of that subject is applicable to the more complex forms which present too much algebraic difficulty for our former method.

51. SUB-TANGENT AND SUB-NORMAL. The distance from the point of tangency to the point where the tangent intersects the X axis is called the length of the tangent, and the projection of this portion of the tangent line on the X axis is called the sub-tangent. The distance from the point of tangency to the point where the normal intersects the X axis is called the length of the normal, and the projection of this portion of the normal line on the X axis is called the sub-normal. Thus if the center of the circle be at the origin, PT, PO, OQ and QT are respectively the lengths of the tangent, normal, sub-normal and sub-tangent at P. Let the radius of the circle be r; the co-ordinates of P be x_1, y_1; and of T be x_2, 0. Then in this special case it is evident that the lengths of the normal and sub-normal are r and x_1. To find the lengths of the tangent and sub-tangent we write the equation of the tangent at P and find its intersection with the X axis. The length of the sub-tangent is then

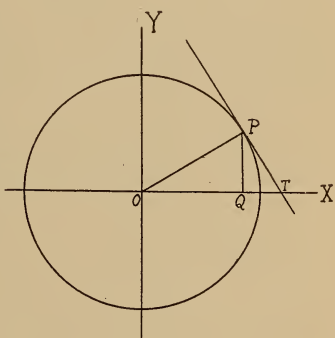

FIG. 19.

$$x_2 - x_1 = \frac{r^2 - x_1^2}{x} = \frac{y_1^2}{x_1},$$

and the length of the tangent is

$$\sqrt{(x_2 - x_1)^2 + y_1^2}$$

$$= \sqrt{\frac{y_1^4}{x_1^2} + y_1^2} = \frac{y_1}{x_1}r;$$

results which might have been directly obtained by the aid of trigonometry.

PROBLEMS.

1. What is the condition of tangency to the circle when the equation of the line is given in the form

$$Ax + By + C = 0?$$

2. Find the equations of the tangents to

$$x^2 + y^2 = 4$$

through (1, 3), (5, 6), (1, —1).

3. Find the tangents to

$$(x-1)^2+(y-2)^2=9$$

through the point (5, 3).

4. Show from the quadratic for determining m that the tangents through P are real or imaginary according as P is outside or inside the circle.

5. Show that the equation of the tangent to

$$x^2+y^2+2ax+2by+c=0$$

at the point (x_1, y_1) is

$$y-y_1=-\frac{x_1+a}{y_1+b}(x-x_1).$$

This form reduces to

$$xx_1+yy_1+ax+by=x_1^2+y_1^2+ax_1+by_1$$

Adding ax_1+by_1+c to both sides, the second member vanishes (why?) and we have a frequently used form

$$xx_1+yy_1+a(x+x_1)+b(y+y_1)+c=0.$$

6. Show that the equation of the normal to the circle

$$x^2+y^2=r^2$$

at the point (x_1, y_1) is

$$xy_1-yx_1=0.$$

Note that the normal to the circle always passes through the center.

52. POLES AND POLARS DEFINED. Given any two tangents to a circle, the chord joining their points of tangency is called the chord of contact. We have just seen that any point determines two tangents to a circle and hence it determines a chord of contact. Similarly any chord of a circle determines the two tangents at its points of intersection with the circle, and hence it determines a point, the intersection of the two tangents. In other words there exists a one to one correspondence between the chords of any circle and the points of the plane, so that to each point there corresponds a single chord and conversely. The chord is called the polar line or merely the polar of the point with respect to the circle, and the point is called the pole of the line with respect to the circle.

53. EQUATION OF THE POLAR. The equation of the polar of the point (x_1, y_1) with respect to the circle

$$x^2 + y^2 = r^2$$

might be derived directly by finding the points of contact of the tangents from (x_1, y_1) and writing the equation of the line through these two points; but the algebraic work is somewhat complicated and we will accordingly make use of a method similar to that of paragraph 38.

Let P be any point (x_1, y_1) and let the two tangents from P to the circle

$$x^2 + y^2 = r^2$$

touch the circle at the points A and B, (x_2, y_2) and (x_3, y_3). Since PA is tangent at (x_2, y_2) its equation is

$$xx_2 + yy_2 = r^2$$

Similarly the equation of PB is

$$xx_3 + yy_3 = r^2.$$

But each of these lines passes through (x_1, y_1) and therefore we have

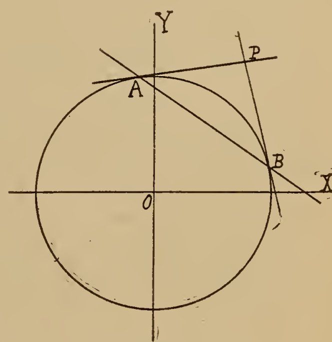

FIG. 20.

$$x_1x_2 + y_1y_2 = r^2$$
$$x_1x_3 + y_1y_3 = r^2.$$

The problem before us is to find an equation of the first degree in x and y which is satisfied when x and y are replaced either by x_2 and y_2 or by x_3 and y_3. An inspection of the pair of equations last written shows that

$$xx_1 + yy_1 = r^2$$

is such an equation. It is therefore the equation of the polar of the point (x_1, y_1) with respect to the circle

$$x^2 + y^2 = r^2.$$

When the point (x_1, y_1) is on the circle the equation of the polar becomes the equation of the tangent at (x_1, y_1). In other words the tangent is only a special case of the polar, being the polar of the point of tangency. By allowing the point P in the figure to approach the circle the student can convince himself that the tangent is the limiting position of the polar as the pole approaches the circle.

The discussion given above in no way depends on the location of the point P outside the circle. If P is inside the circle the two tangents are imaginary and the points of contact also imaginary; but if the pole P and the circle are real the points of contact are conjugate imaginary points and the polar is real, a fact which is also evident from the equation.

54. CO-ORDINATES OF THE POLE. To find the pole of a given line with respect to the circle

$$x^2 + y^2 = r^2$$

we may also use a method shorter than the direct one of finding the intersection of the two tangents having the given line as chord of contact. Let the given line be

$$ax + by + c = 0$$

and assume the co-ordinates of its pole to be (x_1, y_1). Then the equation of the line must be

$$xx_1 + yy_1 = r^2.$$

Since these two equations represent the same line their coefficients must be proportional (why not equal?) i. e.,

$$\frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c}$$

whence $$x_1 = \frac{-ar^2}{c} \qquad y_1 = \frac{-br^2}{c}$$

55. POLAR AS LOCUS OF HARMONIC CONJUGATES. Consider any point P and draw through P a line meeting the circle in the points Q and R. Let S be the harmonic conjugate of P with respect to Q and R. The locus of S as the line rotates through P is the polar of P with respect to the circle.

The proof of this theorem assumes the following:

(1) Problem 7, paragraph 9;

(2) The roots of $cx^2 + bx + a = 0$ are the reciprocals of the roots of $ax^2 + bx + c = 0$;

(3) The sum of the roots of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$.

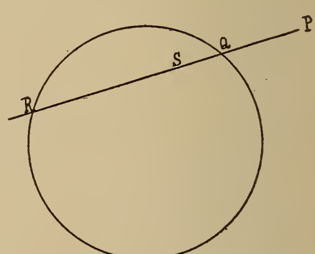

FIG. 21.

Let the co-ordinates of P be (x_1, y_1) and the equation of the circle be

$$x^2 + y^2 = r^2$$

Write the equation of the line through P in the form

$$x = x_1 + \rho \cos \alpha$$
$$y = y_1 + \rho \cos \beta$$

Substitute these values of x and y in the equation of the circle, and the distances PQ and PR from the point P to the intersections of the line and the circle are given by the quadratic

$$\rho^2 (\cos^2 \alpha + \cos^2 \beta) + 2\rho (x_1 \cos \alpha + y_1 \cos \beta) + x_1^2 + y_1^2 - r^2 = 0$$

and
$$\frac{1}{PQ} + \frac{1}{PR} = -\frac{2(x_1 \cos \alpha + y_1 \cos \beta)}{x_1^2 + y_1^2 - r^2}$$

Let the co-ordinates of S be (x', y') then

$$PS = \sqrt{(x' - x_1)^2 + (y' - y_1)^2}$$

The necessary and sufficient condition that S shall be the harmonic conjugate of P with respect to Q and R, i. e., the equation of the locus of S, is therefore

$$\frac{-2(x_1 \cos \alpha + y_1 \cos \beta)}{x_1^2 + y_1^2 - r^2} = \frac{2}{\sqrt{(x_1 - x')^2 + (y_1 - y')^2}}$$

But
$$\cos \alpha = \frac{x' - x_1}{\sqrt{(x' - x_1)^2 + (y' - y_1)^2}},$$

$$\cos \beta = \frac{y' - y_1}{\sqrt{(x' - x_1)^2 + (y' - y_1)^2}}$$

Substituting these values we have for the equation of the locus

$$x'x_1 + y'y_1 = r^2$$

i. e.,
$$xx_1 + yy_1 = r^2$$

as before.

56. POLAR AS LOCUS OF POLES. An important problem here presents itself. Given a fixed point and a circle, to find the locus of the poles with respect to the given circle of all lines through the given point. This locus is evidently a definite curve and therefore its equation must be a single relation between the co-ordinates of the variable point and known constants, but without arbitrary parameters. Let the point be (x_1, y_1) and the circle be given in the form

$$x^2 + y^2 = r^2.$$

The equation of any line through the point is

$$y - y_1 = m(x - x_1)$$

and the co-ordinates of the pole are

$$x' = \frac{-mr^2}{-mx_1 + y_1} \qquad y' = \frac{r^2}{-mx_1 + y_1}$$

We have here two equations connecting the co-ordinates of the variable point with known constants and with the arbitrary parameter m. What we desire is a single relation free from arbitrary parameters, connecting the co-ordinates with each other and with known constants. We therefore eliminate m and find for the equation of the desired locus (after dropping accents)

$$xx_1 + yy_1 = r^2$$

showing that the locus of the poles of all lines through the point (x_1, y_1) is the polar of (x_1, y_1).

PROBLEMS.

1. Find the polars of the following points with respect to

$$x^2 + y^2 = r^2$$

$$(1, 3),\ (-2, 4),\ (k, -p),\ (0, a),\ (b, \sin k),\ (c, \frac{\pi}{2}).$$

2. Find the poles of the following lines with respect to

$$x^2 + y^2 = 10.$$

$$3x - 2y + 4 = 0 \qquad \frac{x}{a} + \frac{y}{b} = 1$$

$$3x - 7y + 4 = 0 \qquad x - y = 0$$

$$y = mx + b \qquad y = 4$$

$$\begin{cases} x = 3 + 2r \\ y = 5 + 6r \end{cases}$$

3. Find the general equation of the polar of a point on the X axis with respect to the circle

$$x^2 + y^2 = r^2;$$

of a point on the line $x = 2$; on the line $x = 3y$; on the line $x = 3y + 2$.

4. Given two diameters at right angles to each other show that the polars of all points on one are parallel to the other, and conversely that the poles of all lines parallel to one lie on the other.

5. Show that if the point (x_1, y_1) lies on its own polar with respect to

$$x^2 + y^2 = r^2$$

it lies also on the circle.

6. Show that the condition which must be satisfied in order that (x_1, y_1) may lie on the polar of (x_2, y_2) is identical with the condition which must be satisfied in order that (x_2, y_2) may lie on the polar of (x_1, y_1), and thus prove the following theorem.

Let A and B be two points. Then if A lies on the polar of B, B lies on the polar of A.

7. On the basis of the theorem just stated deduce a method of constructing with ruler and compass the pole of a given line which does not meet the circle in real points.

8. Construct with ruler and compass the polar of a point inside the circle.

9. Show both geometricly and algebraicly that the polar of the center of the circle is the line at infinity.

CHAPTER XI.

ADDITIONAL WORK ON THE SUBJECT OF LOCI.

57. **GENERAL REMARKS ON LOCI PROBLEMS.** Problems in which the restrictions on the movement of the tracing point are given and the equation of the locus demanded are all alike in the fact that the method of solution consists merely in the translation of the law of movement of the tracing point into algebraic language. They may, however, be divided into two general classes. In the first class fall problems of the type discussed in paragraph 24, in which the statement of the law gives the locus immediately. In the second class fall problems of the type discussed in paragraphs 55 and 56, in which the attempt to translate the law of movement of the point leads to relations connecting the co-ordinates of the tracing point with each other and with certain arbitrary parameters.

In every legitimate locus problem in plane geometry the number of equations expressing such relations, either between the variable co-ordinates and the parameters, or between the parameters themselves, is always one more than the number of the parameters, so that it is possible by the elimination of the parameters to deduce a single relation connecting the co-ordinates of the tracing point with each other and with known constants, i. e., the equation of the locus. If in any particular case the number of equations is less than this, one of two things must be true. Either the conditions laid down do not force the tracing point to follow a definite path, or the student has failed to impose on the co-ordinates of the point or on the parameters all of the limitations imposed by the problem.

As illustrations of what has been said above consider the solutions of the following:

1. A line of fixed length slides along the co-ordinate axes, keeping one end on each axis. Find the locus of its middle point.

The point P, (x, y), has its position determined by the two variable quantities a and b. If k is the fixed length of the line we have as the algebraic translation of the restrictions on the movement of P the following equations:

$$x = \frac{a}{2}, \quad y = \frac{b}{2},$$
$$a^2 + b^2 = k^2.$$

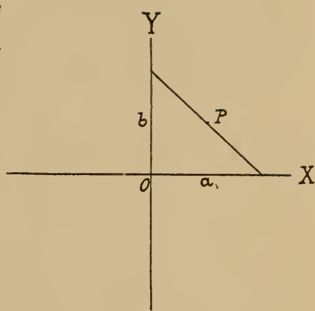

FIG. 22.

From these three equations we eliminate the two parameters a and b, and deduce the single relation

$$4x^2 + 4y^2 = k^2$$

the equation of the desired locus.

2. Given a fixed point on a circle and a variable chord through that point. Find the locus of the point which divides the chord in the ratio $\frac{m}{n}$.

Let O be the fixed point, OP any position of the variable chord, and Q the point whose locus we desire to find. We are free to locate our axes in any position, and in order that the work may be as simple as possible we select the radius of the circle through O as the X axis and the tangent at O as the Y axis. The circle then has $(r, 0)$ as its center and its equation is

$$x^2 - 2rx + y^2 = 0.$$

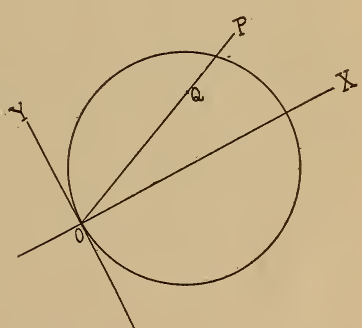

FIG. 23.

The variation of Q is evidently produced by the variation of P along the circle. Let P be (x_1, y_1) and Q (x', y') and we have at once as the translation into algebraic language of the limitations on the movement of Q

$$x' = \frac{mx_1}{m+n} \qquad y' = \frac{my_1}{m+n}$$

two relations embracing x', y' and the two variable parameters x_1 and y_1, one relation less than we need. It is evident that the

limitations force Q to trace a definite locus, we must therefore have failed to impose one of the limitations of the problem. Searching for this omitted limitation we soon see that we have not restricted (x_1, y_1) to the circle. Imposing this restriction we have our desired third relation

$$x_1^2 - 2rx_1 + y_1^2 = 0,$$

which combined with the two already found enables us to eliminate m and n and deduce the desired relation between x' and y',

$$\left(\frac{m+n}{m}\right)x'^2 - 2rx' + \left(\frac{m+n}{m}\right)y'^2 = 0$$

or, dropping accents,

$$\left(\frac{m+n}{m}\right)x^2 - 2rx + \left(\frac{m+n}{m}\right)y^2 = 0.$$

PROBLEMS.

1. A, B, C are three fixed points on a straight line and P a variable point subject to the condition

$$\text{angle } APB = \text{angle } BPC.$$

Find the locus of P.

(In this as well as the other problems of this list the student will find it well to locate his axes in such a way as to give the greatest possible simplicity to the work without destroying the generality of the problem.)

2. Through a fixed point O on a circle chords are drawn and on each chord, extended, a point P is taken such that OP is twice the length of the chord. Find the locus of P.

3. Subject the general circle

$$(x-a)^2 + (y-b)^2 = r^2$$

to the condition of passing through the two fixed points (x_1, y_1) and (x_2, y_2) and find the locus of the center.

4. Find the locus of the point from which a fixed segment AB on a given line subtends a right angle.

5. The two tangents from a variable point to a fixed circle make with each other a constant angle. Find the locus of the variable point.

6. The distance of the point P from a fixed point on a circle centered at the origin is equal to the slope of the polar of P with respect to the circle. Find the locus of P

7. A and B are two fixed points. The distance of P from A equals the cosine of the angle PAB. Find the locus of P.

8. The distance of P from its polar with respect to a given circle centered at the origin is equal to the slope of the polar. Find the locus of P.

9. A line moves parallel to its original position. On the line a point P is taken so that the distance from P to the Y axis is equal to the distance from P to the point where the line meets the X axis. Find the locus of P.

10. A line of constant length slides on the co-ordinate axes, keeping one extremity on each axis. Find the locus of its pole with respect to a given circle centered at the origin.

11. The distance of the point P from its polar with respect to a fixed circle centered at the origin is equal to the sum of the intercepts of the polar on the co-ordinate axes. Find the locus of P.

12. A is a fixed point outside and Q a variable point on the circumference of a fixed circle. Find the locus of the point on the line AQ whose distance ratio with respect to A and Q is 3.

13. Find the locus of the poles, with respect to a given circle, of a system of parallel straight lines.

14. A variable line is subject to the condition that it must be tangent to

$$x^2 + y^2 = 16.$$

Find the locus of its pole with respect to

$$x^2 + y^2 = 4.$$

15. The line joining the point P to a fixed point on the circumference of a given circle is perpendicular to the polar of P with respect to the same circle. Find the locus of P.

16. Which of the loci deduced above are circles?

CHAPTER XIII.

THE GENERAL EQUATION OF THE SECOND DEGREE.

58. NATURE OF THE PROBLEM AND OF THE METHOD EMPLOYED. We take up now the study of the general conic as represented by the general equation of the second degree, a special case of which has been studied in the chapter on the circle. We shall show that every conic has two axes of symmetry and that by making the axes of co-ordinates coincident with or parallel to these axes of symmetry all equations of the second degree are reduced to three type forms. These type forms will then be investigated in much the same manner as the circle was investigated in Chapter XI.

For the first part of the investigation we shall use the parametric form of the equations of a straight line, developed in paragraph 39, which bring into evidence a fixed point (x_1, y_1), the direction of a line through the point, and the distance measured along the line from the fixed point to a variable point. By putting the variable point on the conic and rotating the line we shall be able to investigate the curve by noting the changing value of the distance from the fixed point to the variable point on the conic, in much the same manner that the bottom of a lake is investigated by measurement of its depth at various points.

59. THE "r" EQUATION. Consider any point (x_1, y_1) and write any line through it in the form

$$x = x_1 + lr$$
$$y = y_1 + mr$$

where l and m are the cosines of the angles which the line makes with the X and Y axes. To find the distances along the line from the point (x_1, y_1) to the conic we substitute x and y, as given by the equations of the line, in the general equation of the second degree

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

This gives us

$$\begin{aligned}&(ax_1^2+by_1^2+2hx_1y_1+2gx_1+2fy_1+c)\\&\quad+2r[l(ax_1+hy_1+g)+m(hx_1+by_1+f)]\\&\quad+r^2(al^2+bm^2+2hlm)=0\end{aligned}$$

an equation in r whose roots are the distances from the fixed point to the conic. Since the equation is a quadratic there are two such distances and we have the theorem:

A conic is met by any straight line in two points.

60. ONE CHORD IS BISECTED AT ANY POINT.

If the point (x_1, y_1) is so placed as to be midway between the points of intersection of the line and the conic the two values of r given by the r equation are equal in value and opposite in sign. The necessary and sufficient condition for this is the vanishing of the coefficient of the first degree term in r, i. e.,

$$l(ax_1+hy_1+g)+m(hx_1+by_1+f)=0. \qquad \text{A}$$

This equation may be satisfied in several ways. First, x_1 and y_1 may be given any arbitrary values and the equation satisfied by proper choice of $\frac{m}{l}$, i. e., by giving the proper direction to the line. Note that only one value of $\frac{m}{l}$ will satisfy the equation. From this method of satisfying the condition we derive the theorem:

Through any point in the plane there may be drawn one and in general but one chord of a given conic which is bisected at that point.

61. CENTER OF A CONIC.

Again, it is always possible to find one and only one point which satisfies both of the equations

$$\begin{aligned}ax+hy+g&=0\\hx+by+f&=0\end{aligned}$$

If this point is taken as (x_1, y_1) the coefficient of r is zero without regard to the value of $\frac{m}{l}$, i. e., the chord is bisected at (x_1, y_1) no matter what its direction. This gives us the second theorem:

Every conic has a center of symmetry whose co-ordinates are

determined by the equations

$$ax_1 + hy_1 + g = 0$$
$$hx_1 + by_1 + f = 0$$

62. DIAMETERS OF A CONIC. Again, let $\frac{m}{l}$ have a fixed value. It is possible to satisfy the condition A by choice of (x_1, y_1). In fact if we regard $\frac{m}{l}$ as fixed and x_1 and y_1 as variable the condition becomes the equation of a straight line, which is evidently the locus of the middle points of a system of parallel chords with the slope $\frac{m}{l}$. Such a locus is called a diameter of the conic. It is easy to see that all diameters pass through the center of the conic. We have now our third theorem:

The equation of the diameter bisecting the family of chords whose slope is $\frac{m}{l}$ is

$$l(ax + hy + g) + m(hx + by + f) = 0$$

63. AXES OF SYMMETRY OF A CONIC. An axis of symmetry differs from other diameters in that it is perpendicular to the chords it bisects. Given a family of chords of slope $\frac{m}{l}$ the corresponding diameter has the slope

$$-\frac{al + hm}{hl + bm}$$

If this diameter is to be perpendicular to the chords it bisects we must have

$$\left(\frac{m}{l}\right)\left(-\frac{al + hm}{hl + bm}\right) = -1$$

i. e.,

$$alm + hm^2 = hl^2 + blm$$

or

$$\left(\frac{m}{l}\right)^2 + \left(\frac{a - b}{h}\right)\frac{m}{l} - 1 = 0.$$

The roots of this equation are the slopes of systems of chords perpendicular to the diameter which bisects them. The equation is a quadratic, therefore there are two such systems. The product of the two roots is -1, therefore the two systems are perpendicular to each other. We have now our fourth theorem:

Every conic has two axes of symmetry, and these two axes are perpendicular to each other.

64. REDUCTION OF THE GENERAL EQUATION. The work of the last article enables us to determine the angles which the two axes of symmetry make with the X axis. Let us assume that the axes of co-ordinates have been made parallel to the axes of symmetry by rotation through one of these angles, and let the form of the equation referred to these new axes be

$$Ax^2 + By^2 + 2Hxy + 2Gx + 2Fy + C = 0. \qquad (1)$$

Two questions are now before us. This particular choice of axes must entail certain values for some of the coefficients of the equation or, what amounts to the same thing, special relations between them. The determination of these values or relations is our first question, and the determination of the distance from the axes of co-ordinates to the axes of symmetry is the second.

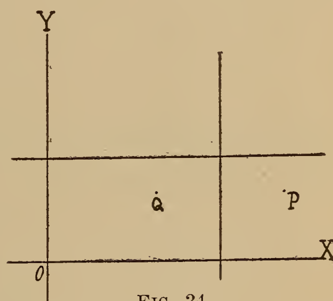

FIG. 24.

Let AB and CD (Fig. 24) be the axes of symmetry and let their distances from the axes of co-ordinates be k and l (equations $y - l = 0$, $x - k = 0$). The necessary and sufficient condition that

$$x - k = 0$$

may be an axis of symmetry is that if any point P_1, co-ordinates $(k + r, y_1)$, be on the curve, the point Q, co-ordinates $(k - r, y_1)$, shall also be on the curve, i. e., if

$$A(k+r)^2 + By_1^2 + 2H(k+r)y_1 + 2G(k+r) + 2Fy_1 + C = 0$$

so also $A(k-r)^2 + By_1^2 + 2H(k-r)y_1 + 2G(k-r) + 2Fy_1 + C = 0$

Subtracting the second from the first and dividing by $4r$ (What right have we to divide by r?) we have

$$Ak + G + Hy_1 = 0$$

an equation of the first degree in y_1 which is satisfied by the y co-ordinate of any point on the conic. This can be true in only two ways; either the general conic must consist of straight lines, or the equation last written is not a condition but an identity. If the former of these alternatives were true the general equation must split up into first degree factors, but it does not; we have

therefore $\qquad Ak + G + Hy_1 \equiv 0$

i.e., $\qquad Ak + G = 0 \quad \text{and} \quad H = 0.$

Similarly treating the other axis of symmetry we have

$$Bl + F = 0 \quad \text{and} \quad H = 0.$$

If then the axes of co-ordinates are taken parallel to the axes of symmetry of the conic the equation has no term in xy, and the coefficients A, B, F, G, are connected with each other and with the distances from the axes of co-ordinates to the axes of symmetry by the relations

$$k + \frac{G}{A} = 0 \qquad l + \frac{F}{B} = 0\,*$$

(The point (k, l) is evidently the center of symmetry, and its co-ordinates might therefore have been found by the method of article 61.)

Apply the transformation

$$x = x' - \frac{G}{A}$$

$$y = y' - \frac{F}{B}$$

and the new axes of co-ordinates coincide with the axes of symmetry of the conic, while the equation reduces to the form

$$(1) \qquad Ax^2 + By^2 + K = 0$$

in which K denotes the new constant term. If either A or B is zero (both cannot be, problem 4, article 33) the above transformation cannot be made, since in this case the center of symmetry and one of the axes of symmetry are at infinity. The simplification of

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$$

must therefore in this case be accomplished in some other way. Let A be the coefficient that vanishes. There is nothing to prevent our applying the transformation

$$y = y' - \frac{F}{B}$$

which makes the X axis coincide with that axis of symmetry which

---

*i. e., the special choice of axes made at the beginning of this paragraph leaves (1), not in the general form there written, but in the special form

$$Ax^2 + By^2 - 2kAx - 2lBy + C = 0.$$

is in the finite part of the plane. This transformation reduces the equation to the form

$$By^2 + 2Gx + L = 0$$

where L is the new constant term. This curve crosses the X axis at the point $(-\frac{L}{2G}, 0)$ which is in the finite part of the plane so long as G does not vanish. We therefore apply the transformation

$$x = x' - \frac{L}{2G}$$

which moves the Y axis to this point of intersection and reduces the equation to the form

$$(2) \qquad By^2 + 2Gx = 0.$$

If G vanishes the equation

$$By^2 + 2Gx + L = 0$$

reduces at once without transformation to

$$(3) \qquad By^2 + L = 0.$$

We have now succeeded in showing that each and every equation of the second degree in two variables may, by a mere transformation of co-ordinates, be reduced to one of the types (1), (2), (3) above. (3) reduces at once to

$$y = \pm\sqrt{-\frac{L}{B}}$$

representing two real or imaginary lines parallel to the X axis. We may therefore dismiss it from further consideration. (1) and (2) remain to be studied. Since all the conics reducible to (1) have their centers in the finite part of the plane we shall frequently refer to them as central conics.

CHAPTER XIV.

THE ELLIPSE AND THE HYPERBOLA.

65. **DETERMINATION OF FORM.** We take up now the consideration of type (1) of the preceding chapter, the equation

$$Ax^2 + By^2 + K = 0.$$

If K vanishes the equation at once reduces to the form

$$(\sqrt{A}x + \sqrt{-B}y)(\sqrt{A}x - \sqrt{-B}y) = 0$$

and therefore represents two real or imaginary straight lines through the origin. If K does not vanish and A, B, and K have all the same sign, the equation cannot be satisfied by any real point (sum of three positive or three negative quantities cannot be zero); and since we are for the present interested only in real loci we shall give no further attention to this case. When the signs are not all the same divide by K, put $-\frac{A}{K} = \alpha$, $-\frac{B}{K} = \beta$ and the equation reduces to the form

$$\alpha x^2 + \beta y^2 = 1$$

This curve meets the X axis in the points $\left(\frac{1}{\sqrt{\alpha}}, 0\right)$ and $\left(-\frac{1}{\sqrt{\alpha}}, 0\right)$ and the Y axis in the points $\left(0, \frac{1}{\sqrt{\beta}}\right)$ and $\left(0, -\frac{1}{\sqrt{\beta}}\right)$, and the lengths of the segments determined by the curve on the X and Y axes (and hence on the axes of symmetry) are $\frac{2}{\sqrt{\alpha}}$ and $\frac{2}{\sqrt{\beta}}$. These values might logically be called the lengths of the axes of the curve, but as one or the other of the quantities α, β may be negative mathematicians have agreed to define the lengths of the axes as the moduli (see appendix E) of these values, i. e., $\left|\frac{2}{\sqrt{\alpha}}\right|$, $\left|\frac{2}{\sqrt{\beta}}\right|$, and in this way avoid the introduction of imaginary lengths. If we denote the semi-axes thus defined by a and b we have

$$2a = \left|\frac{2}{\sqrt{\alpha}}\right| \qquad 2b = \left|\frac{2}{\sqrt{\beta}}\right|$$

If α and β are both positive we have

$$\alpha = \frac{1}{a^2} \qquad \beta = \frac{1}{b^2}$$

and the equation takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solve this equation for y in terms of x and the truth of the following statements is at once evident. As x increases in numerical value, or, to say the same thing more technically, as $|x|$ increases, $|y|$ decreases from the

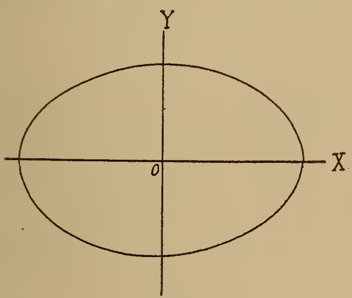

FIG. 25.

value b which it has when $|x|$ is zero to the value zero which it has when $|x|$ is a. As $|x|$ increases beyond a, $|y|$ increases indefinitely, but since y is imaginary for these values the corresponding points are not represented in the plane. The curve therefore lies wholly within the rectangle formed by the lines

$$x \pm a = 0 \text{ and } y \pm b = 0$$

Careful plotting will show it to be of the form here given. It is called an ellipse.

If one, let us say β of the quantities α, β is negative, we have

$$\alpha = \frac{1}{a^2} \qquad -\beta = \frac{1}{b^2}$$

and the equation takes the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Solve this equation for y in terms of x and the truth of the following statements is at once evident. As $|x|$ increases, beginning at zero, $|y|$ decreases from the value b, which it has when $|x|$ is zero, to the value zero, which it has when $|x|$ is a.

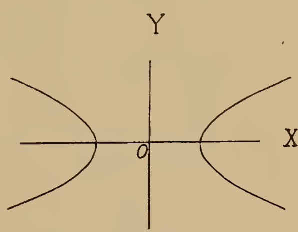

FIG. 26.

Over this range, however, y is imaginary so the corresponding points are not represented in the plane. As $|x|$ increases beyond a, $|y|$ increases indefinitely. The curve therefore lies wholly without the lines

$$x \pm a = 0$$

and extends upward and downward indefinitely. Careful plotting will show it to be of the form here given. It is called an hyperbola.

In plotting this diagram a was assumed greater than b. If a is less than b, the conic is turned along the other axis. The question of size is evidently the important one in distinguishing between the axes and they are therefore spoken of as major and minor. In the development of the theory of the ellipse we shall assume that a denotes the length of the semi-major axis.

In plotting this diagram β was assumed negative. If a is negative the conic is turned along the other axis. The important question here is not one of size but of the character (real or imaginary) of the points in which the conic meets the axes. The axis met by the conic in real points is called the transverse axis, the one met in imaginary points is called the conjugate axis. In the development of the theory of the hyperbola we shall assume that a denotes the length of the semi-transverse axis.

66. EARLY GEOMETRIC DEFINITIONS. These curves were well known to geometricians before the invention of analytic geometry, and each of them had its geometric definitions. Two of these are as follows:

(A) An ellipse | An hyperbola

is the locus of a point the

sum | difference

of whose distances from two fixed points, called the foci, is constant.

(B) An ellipse | An hyperbola

is the locus of a point whose distance from a fixed point divided by its distance from a fixed line is a constant

less | greater

than unity.

The fixed point is called the focus, the fixed line the directrix, and the constant ratio the eccentricity.

If we attempt to deduce the equations of these curves directly from the definitions just given the resulting forms will depend upon the choice of co-ordinate axes. In deducing the equations from definition (A) we choose the line joining the two foci as the X axis and the perpendicular bisector of the segment between the

foci as the Y axis. Then if the distance between the foci be taken as $2c$ the foci are $(-c, 0)$ and $(c, 0)$ and our definitions lead at once to the equations

$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2k$	$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2k$

where $2k$ is the constant

sum	difference

mentioned in the definition. Rationalize and reduce and each of the forms leads to the single equation

$$\frac{x^2}{k^2} + \frac{y^2}{k^2 - c^2} = 1$$

If we compare this with our standard forms we have

$a^2 = k^2$	$a^2 = k^2$
$b^2 = k^2 - c^2$	$-b^2 = k^2 - c^2$
$= a^2 - c^2$	$= a^2 - c^2$
$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$

and we have as foci the two real points

$(\sqrt{a^2 - b^2}, 0)$	$(\sqrt{a^2 + b^2}, 0)$
$(-\sqrt{a^2 - b^2}, 0)$	$(-\sqrt{a^2 + b^2}, 0)$

Note that the foci were assumed on the X axis and the resulting equation identified with one for which the X axis is the

major	transverse

axis of symmetry. Hence we may say that the foci are two real points on the

major	transverse

axis of the conic and symmetrically situated with respect to the other axis.

If on the other hand, the foci are assumed on the Y axis the resulting equation is

$$\frac{x^2}{k^2 - c^2} + \frac{y^2}{k^2} = 1$$

Identifying this with the same equations as before, we have

$b^2 = k^2$	$-b^2 = k^2$
$a^2 = k^2 - c^2$	$a^2 = k^2 - c^2$
$= b^2 - c^2$	$= -b^2 - c^2$
$c^2 = b^2 - a^2$	$c^2 = -a^2 - b^2$

and we have as foci the two imaginary points

$(0, \sqrt{b^2 - a^2})$	$(0, \sqrt{-a^2 - b^2})$
$(0, -\sqrt{b^2 - a^2})$	$(0, -\sqrt{-a^2 - b^2})$

Note that in this case the foci are assumed on the Y axis and the resulting equation identified with one for which the Y axis is the

minor	conjugate

axis of symmetry. Hence we may say that the foci are two imaginary points on the

minor	conjugate

axis of the conic and symmetrically situated with respect to the other axis.

But since either assumption leads, when c and k are properly determined, to the same equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

it follows that the conic represented by this equation (i. e., any central conic) has four foci, two real on the

major	transverse

axis, and two imaginary on the

minor	conjugate

axis. Hereafter when the foci are referred to it is understood that the reference is to the real foci unless both are mentioned.

In deducing the equations from definition (B) we choose the directrix as the Y axis and the perpendicular let fall upon it from the focus as the X axis. We denote the distance from the focus to the directrix by d and the eccentricity by e. The definition then leads at once to the equation

$$\frac{\sqrt{(x - d)^2 + y^2}}{x} = e$$

or on reduction

$$x^2(1 - e^2) + y^2 - 2dx + d^2 = 0.$$

The absence of the xy term shows that the axes of co-ordinates are parallel to the axes of symmetry, and the absence of the y term shows that the X axis coincides with an axis of symmetry. But the presence of the x term shows that the Y axis does not coincide with an axis of symmetry and a transformation of co-ordinates

must be made before we can compare our equation with the standard forms. We therefore bring the Y axis into coincidence with the other axis of symmetry by the substitution

$$x = x' + \frac{d}{1 - e^2}$$

where $\frac{d}{1 - e^2}$ is the distance from the directrix to the axis of symmetry to which it is parallel. The transformation gives an equation which reduces finally to

$$\frac{x^2}{\frac{d^2e^2}{(1 - e^2)^2}} + \frac{y^2}{\frac{d^2e^2}{1 - e^2}} = 1,$$

an equation of the desired standard form.

For the ellipse e is less than unity and the coefficients of both x^2 and y^2 are positive. Comparing our present equation with the standard form we have

$$a^2 = \frac{d^2e^2}{(1 - e^2)^2}$$

$$b^2 = \frac{d^2e^2}{(1 - e^2)}$$

whence

$$\frac{b^2}{a^2} = 1 - e^2$$

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}$$

Again, replacing e by $\frac{c}{a}$ and $1 - e^2$ by $\frac{b^2}{a^2}$ we have, on solving for d

$$d = \pm \frac{b^2}{c}$$

For the hyperbola e is greater than unity and the coefficient of x^2 is positive while that of y^2 is negative. Comparing our present equation with the standard form we have

$$a^2 = \frac{d^2e^2}{(1 - e^2)^2}$$

$$-b^2 = \frac{d^2e^2}{(1 - e^2)}$$

whence

$$-\frac{b^2}{a^2} = 1 - e^2$$

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{c}{a}$$

Again, replacing e by $\frac{c}{a}$ and $1 - e^2$ by $-\frac{b^2}{a^2}$ we have, on solving for d

$$d = \pm \frac{b^2}{c}$$

The quantity c will hereafter be called the linear eccentricity.

An investigation of the significance of the double sign of d leads to interesting results.

To take the positive sign is to assume that the focus is on the right of the directrix, while the distance which the Y axis must be moved to pass from coincidence with the directrix to coincidence with the axis of symmetry is

positive (e less than unity) and greater than d.	negative (e greater than unity).

To take the negative sign is to assume that the focus is on the left of the directrix, while the distance through which the Y axis is moved is

negative	positive

with the same numerical value as before.

In other words, if we start with a focus on the right of a directrix and move the Y axis a certain distance to the

right	left

we reduce the equation to a certain form. If we start with a focus on the left of a directrix, we have a different set of axes and a different equation. But when we move the Y axis the same distance as before to the

left	right

we reduce the equation to the same form as before, i. e., the two equations given by the two signs of d represent the same curve, but referred to different systems of co-ordinates.

Evidently therefore the two signs of d correspond to two foci and two directrices. Evidently also the directrices are symmetrically situated with respect to one axis of symmetry of the conic and cross the other axis at points

without	within

the segment determined on that axis by the foci. It is not difficult to show that these points are also

without	within

the segment determined on the axis by the intersection of the axis and the conic.

There are of course a pair of directrices corresponding to the pair of imaginary foci. These are however imaginary, a statement whose proof will follow at once from the solution of problem 8, article 71.

67. MECHANICAL CONSTRUCTIONS. Definition (A) leads to a simple mechanical method of constructing an ellipse. Fasten at the two foci the two ends of a cord whose length is the constant sum of the focal distances of the tracing point. Draw the cord to one side with a pencil and draw the pencil along keeping the cord tightly drawn. The resulting curve evidently satisfies the definition of an ellipse.

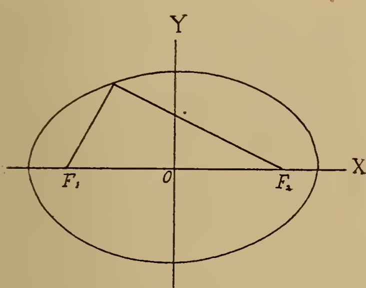

FIG. 27.

Definition (A) leads to a simple mechanical method of constructing an hyperbola. Make a ruler of the form shown in Fig. 28 with the center of the opening P on the straight edge AB extended. By means of this opening P pivot the ruler at one focus F_1. To the other end, B, of the ruler fasten one end of a cord shorter than the ruler by an amount equal to the constant difference of the focal distances of the tracing point and fasten the other end of the cord at the other focus F_2. Rotate the ruler about F_1, keeping the cord pressed tightly against the ruler by a pencil. The resulting curve evidently satisfies the definition of an hyperbola. The other branch is drawn by reversing the apparatus.

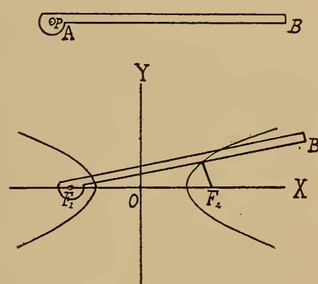

FIG. 28.

PROBLEMS.

1. Let the lengths and positions of the axes of an ellipse and an hyperbola be given. Deduce geometric constructions for the foci and the directrices.

2. Find the eccentricity, lengths of axes, location of foci and location of directrices of the following central conics:

$$3x^2 - 2y^2 + 1 = 0 \qquad 4x^2 + 2y^2 = 12$$
$$-9x^2 + 10y^2 = 144 \qquad 4x^2 + y^2 = 1$$

3. The following conics are assumed to have their centers at the origin and their axes of symmetry as axes of co-ordinates Find their equations.

(1) $e = \frac{1}{2}$ $a = 4.$ (2) $a = 5,$ $c = 2.$

(3) $d = 2,$ $e = 4.$ (4) $a = 3,$ $b = 2.$

(5) $d = 2,$ $e = \frac{1}{4}$ (6) $c = 4,$ $a = 3.$

4. Express the distance between the two directrices in terms of a and e and show that for the ellipse it is greater than $2a$, and for the hyperbola less than $2a$.

5. Let the length of the transverse axis of an hyperbola be constant and let the eccentricity increase indefinitely. Show that under these conditions the directrix tends to coincide with the conjugate axis.

6. To what limiting form does the ellipse tend as a tends to b? What is the limit of the eccentricity?

7. When a tends to b the hyperbola tends to the limiting form represented by

$$x^2 - y^2 = b^2$$

which is called an equilateral hyperbola. What is its eccentricity?

8. Show that as an ellipse tends to a circle the distance of the directrix from the center tends to infinity.

9. If in any conic there be erected at the focus a perpendicular to that axis of symmetry which passes through the focus, the distance between the two points in which this perpendicular meets the curve is called the length of the parameter of the conic. This length is usually denoted by $2p$. If the conic is a central conic referred to its axes of symmetry as co-ordinate axes, the parameter might be defined as the double ordinate through the focus, or as that portion of the line $x + c = 0$ or $x - c = 0$ included between its intersections with the conic. Show that for the ellipse the semi-parameter p is a third proportional to the semi-axes. Show also that $p = a(1 - e^2)$.

10. Determine the corresponding values of p for the hyperbola.

11. Move the axis of Y so that it shall become the tangent at the left hand vertex of the conic. (The vertices of a conic are its

intersections with its axes of symmetry.) The equation of the ellipse then reduces to the form

$$y^2 = \frac{b^2}{a^2}(2ax - x^2)$$

or

$$y^2 = 2\,px(1 - \frac{x}{2a}).$$

Let a and b increase indefinitely, but in such a way that the ratio $\frac{b^2}{a}(=p)$ remains constant. What is the limiting value of ε? What is the limiting form of the second of the two equations just given?

12. Make a similar investigation for the hyperbola.

13. Compare the limiting forms developed in the last two problems with equation (2), article 64, and note that the assumptions just made concerning a and b have the effect of moving the center of the conic to infinity, and therefore correspond to the assumptions which led to equation (2). Hence we may state the theorem: The conic represented by form (2) is the limiting form of both the ellipse and the hyperbola as the eccentricity tends to unity.

14. The lines joining any point on a conic to the foci are called focal radii. Denote their lengths by r and r', let the axes of coordinates be the axes of symmetry, and show that for the hyperbola

$$r = ex - a, \qquad r' = ex + a.$$

15. What are the corresponding values for the ellipse?

68. DIAMETERS. Applying to the special equation of the second degree

$$\alpha x^2 + \beta y^2 = 1$$

the general formulae for the equation of a diameter of a conic developed in article 62, we find that the equation of the diameter bisecting chords of slope m is

$$\alpha x + \beta m y = 0 \qquad \text{i.e.}$$

$b^2x + a^2my = 0$	$b^2x - a^2my = 0$

Given two diameters

$$y = m_1x \text{ and } y = m_2x$$

the work just done shows that the necessary and sufficient condition that the second diameter shall bisect all chords parallel to the first is

$$m_1m_2 = -\frac{\alpha}{\beta} \qquad \text{i. e.,}$$

$m_1m_2 = -\frac{b^2}{a^2}$	$m_1m_2 = \frac{b^2}{a^2}$

From the form of this condition it is evident that if the second diameter bisects all chords parallel to the first, the first also bisects all chords parallel to the second. Such a pair of diameters are said to be conjugate diameters.

Let

$$y = m_1 x \text{ and } y = m_2 x$$

be a pair of conjugate diameters of the conic

$$\alpha x^2 + \beta y^2 = 1$$

and let the points in which they meet the conic be (x_1, y_1), (x_2, y_2), $(-x_1, -y_1)$, $(-x_2, -y_2)$ then

$$m_1 = \frac{y_1}{x_1} \text{ and } m_2 = \frac{y_2}{x_2}$$

and therefore since the diameters are conjugate

$$\frac{y_1 y_2}{x_1 x_2} = -\frac{\alpha}{\beta} \qquad (1)$$

Also since the points are on the conic

$$\alpha x_1^2 + \beta y_1^2 = 1 \qquad (2)$$

and

$$\alpha x_2^2 + \beta y_2^2 = 1 \qquad (3)$$

Solve (2) and (3) for x_1 and y_2 and substitute in (1) whence we have

$$x_2 = \pm\sqrt{\frac{\beta}{\alpha}}\, y_1$$

Substitute this value of x_2 in (1) and we have

$$y_2 = \mp\sqrt{\frac{\alpha}{\beta}}\, x_1$$

If expressed in terms of the lengths of the axes this becomes

$x_2 = \pm \frac{a}{b} y_1,$	$x_2 = \mp i \frac{a}{b} y_1,$
$y_2 = \mp \frac{b}{a} x_1,$	$y_2 = \mp i \frac{b}{a} x_1,$

where the upper sign of x is paired with the upper sign of y. In deciding upon the arrangement of signs for the hyperbola the student must remember that the negative sign is associated with b and that $\frac{1}{i}$ is equal to $-i$.

PROBLEMS.

1. Show that as one of a pair of conjugate diameters of a conic tends to coincide with one axis of symmetry the other tends to coincide with the other axis of symmetry.

2. Show that if one of two conjugate diameters of an hyperbola meets the curve in real points the other meets it in imaginary points, and conversely.

3. Show that two conjugate diameters of an hyperbola are in the same quadrant, and find the angle between them in terms of the semi-axes and the slope of one of the diameters.

4. Make a similar investigation for the ellipse.

5. Let two conjugate diameters of a conic meet it in the points (x_1, y_1) and (x_2, y_2), i.e. in the points (x_1, y_1) and $\left(\sqrt{\frac{\beta}{\alpha}}\, y_1, -\sqrt{\frac{\alpha}{\beta}}\, x_1\right)$, and let the distances of these points from the center be denoted by d_1 and d_2. Then

$$\begin{aligned} d_1^2 + d_2^2 &= x_1^2 + y_1^2 + \frac{\beta}{\alpha} y_1^2 + \frac{\alpha}{\beta} x_1^2. \\ &= \frac{\alpha x_1^2 + \beta y_1^2}{\alpha} + \frac{\alpha x_1^2 + \beta y_1^2}{\beta} \\ &= \frac{1}{\alpha} + \frac{1}{\beta} \end{aligned}$$

In other words the sum of the squares of these distances is constant, i. e., remains unchanged for all positions of the conjugate diameters. In the ellipse both these distances are real, but in the hyperbola one or the other is imaginary. In consequence of this mathematicians agree, as in the case of the axes, to define the length of a semi-diameter of a conic as the modulus of the distance from the center of the conic to the point of intersection of the diameter and the conic. It is usual to denote the lengths of a pair of conjugate semi-diameters by a' and b'. With the above definition in mind, show that the work just done is equivalent to the proof of the theorem:

The sum	The difference

of the squares of the lengths of a pair of conjugate semi-diameters of an

ellipse	hyperbola

is constant and equal to the

sum	difference

of the squares of the lengths of the semi-axes.

6. Find the sines and cosines of the angles made by a pair of conjugate diameters with the axes of an hyperbola, expressing them in terms of a, b, a', b', x_1, y_1.

7. Make a similar investigation for the ellipse.

8. Show that the sine of the angle between any pair of conjugate diameters of an ellipse is $\frac{ab}{a'b'}$.

9. Make a similar investigation for the hyperbola.

10. By aid of the values deduced in problems 6 and 7 find the equations of the ellipse and hyperbola referred to a pair of conjugate diameters as oblique axes of co-ordinates, and show that the equations reduce to the forms

$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$	$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1$

69. SUPPLEMENTAL CHORDS. If any point on a central conic be joined to the extremities of any diameter two chords are formed which are called supplemental chords. If the diameter is the

major	transverse

axis they are called principal supplemental chords. Let (x_1, y_1) be any point on a central conic and (x_2, y_2) one of the extremities of any diameter. Then if the slopes of the two supplemental chords determined at (x_1, y_1) by this diameter are m_1 and m_2

$$m_1=\frac{y_1-y_2}{x_1-x_2} \qquad m_2=\frac{y_1+y_2}{x_1+x_2}$$

and
$$m_1 m_2=\frac{y_1^2-y_2^2}{x_1^2-x_2^2}.$$

But the conditions which must be satisfied in order that (x_1, y_1) and (x_2, y_2) may be on the curve give on subtraction

$$\frac{y_1^2-y_2^2}{x_1^2-x_2^2}=-\frac{\alpha}{\beta}$$

hence
$$m_1 m_2=-\frac{\alpha}{\beta}.$$

i. e., the condition which must be satisfied by the slopes of two supplemental chords is the same which must be satisfied by the slopes of two conjugate diameters. Therefore if one of two supplemental chords is parallel to a diameter, the other is parallel to the conjugate diameter.

PROBLEMS.

1. Given a central conic and a diameter, construct with ruler and compass the conjugate diameter, first from the definition of conjugate diameters and second by aid of a pair of supplemental chords.

2. Given a central conic, find its center with ruler and compass.
3. Given a central conic and a diameter, construct by aid of this diameter a pair of supplemental chords perpendicular to each other.
4. Given a central conic construct its axes with ruler and compass.

PROBLEMS.

70.
TANGENTS AND NORMALS.

1. Following the analogy of the work done in article 49, show that the condition that the line

$$y = mx + h$$

shall be tangent to the conic

$$\alpha x^2 + \beta y^2 = 1$$

is $h = \pm\sqrt{a^2m^2 + b^2}$ | $h = \pm\sqrt{a^2m^2 - b^2}$

Note that after the condition of tangency has been imposed on the equation

$$y = mx + h$$

it contains only one arbitrary parameter. It follows therefore that the tangents which can be drawn to any central conic form a single infinity of lines.

2. Following the method employed in article 50, develop the equation whose roots are the slopes of the tangents from (x_1, y_1) to a central conic. Hence show that if the point is not on the conic, two tangents to the conic can be drawn through it.

3. Following the method of the latter part of article 50, show that the equation of the tangent to the conic

$$\alpha x^2 + \beta y^2 = 1$$

at the point (x_1, y_1) on the conic is

$$\alpha x_1 x + \beta y_1 y = 1$$

State the equations also in terms of a and b for the ellipse and hyperbola.

4. Find the equations of the normals at the same point.

5. Show that the lengths of the sub-tangent and sub-normal for the point (x_1, y_1) are

$$\frac{1 - \alpha x_1^2}{\alpha x_1} \quad \text{and} \quad \frac{\alpha}{\beta} x_1.$$

6. Examine the following lines for cases of tangency to

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \text{or} \quad \frac{x^2}{9} - \frac{y^2}{4} = 1$$

$y + 3x - 4 = 0$	$y = 2$
$x - 4y + 3 = 0$	$5x - 3y + 3 = 0$
$3y - 4x - 12 = 0$	$x - 3y + 5 = 0$

7. Show that the two tangents which may be drawn to any central conic from a given point are real or imaginary according as the point is outside or inside the conic.

8. Find the equations of the tangents from the point (3, 2) to the conics of problem 6.

9. Show that the tangent and normal at any point bisect the angles formed by the focal radii at that point.

10. From the results of the last problem derive a geometric construction for the tangent and normal at any point on a central conic.

11. Find the locus of all points from which the two tangents to a central conic are perpendicular, and construct the locus with ruler and compass. Is the construction always possible in the case of the hyperbola?

12. Find the locus of the feet of the perpendiculars let fall from either focus on the tangents to an ellipse.

(Take $y = mx + h$ equation of straight line.

$h = \pm\sqrt{a^2m^2 + b^2}$ condition of tangency.

$y = -\left(\frac{1}{m}\right)(x - c)$ perpendicular to first line through focus. From these equations eliminate h and m, and derive the desired locus in the form

$$y^2 = x(c - x) \pm \sqrt{a^2(c - x)^2 + b^2y^2}.$$

Rationalize, replace b by its value in terms of a and c, arrange the equation according to powers of c, and the equation reduces to

$$(x^2 + y^2 - a^2)(x^2 + y^2 - 2cx + c^2) = 0$$

The locus is evidently degenerate, representing a circle concentric with the conic and a pair of imaginary lines intersecting at the focus. The question whether the real and imaginary parts of this locus correspond to the perpendiculars let fall upon real and imaginary tangents is left for the investigation of the student.)

13. What changes must be made in the above investigation in order to make it hold for the hyperbola?

14. Show that the product of the two perpendicular distances from the two foci of a central conic to the tangent at (x_1, y_1) is

$$\frac{-b^4a^2e^2x_1^2 + a^4b^4}{b^4x_1^2 + a^4y_1^2}$$

Eliminate y_1 by virtue of the fact that (x_1, y_1) is on the conic, replace e by its value in terms of a and b and thus reduce the value of the product to b^2.

15. Show that the tangents at the extremities of any diameter are parallel to the conjugate diameter.

16. By aid of the last problem and problem 8, article 68, show that the area of the parallelogram formed by the tangents at the extremities of any pair of conjugate diameters of a central conic is independent of the position of the diameters and equal to the area of the rectangle formed by the tangents at the extremities of the axes.

71. POLES AND POLARS. Follow the methods used in the investigation of poles and polars of the circle, and solve the following:

PROBLEMS.

1. Find the equation of the polar of (x_1, y_1) with respect to the conic

$$\alpha x^2 + \beta y^2 = 1.$$

2. Find the co-ordinates of the pole of

$$ax + by + c = 0$$

with respect to the conic

$$\alpha x^2 + \beta y^2 = 1.$$

3. Show that if the pole of a given line with respect to a given central conic is on the conic the polar is the tangent at the pole.
4. Show that if a point lies on its own polar with respect to a central conic, it lies also on the conic.
5. Show that if the polar of a point A passes through B the polar of B passes through A.
6. Given any pair of conjugate diameters of a central conic show that the polars of all points on the one are parallel to the other.
7. Give geometric constructions for poles and polars in the case of both ellipse and hyperbola.
8. Show that the directrices are the polars of the foci.
9. If it can be shown that any investigation does not depend on the rectangularity of the axes, the results of the investigation hold good for oblique axes. Show in this way that the form of the equation of the polar developed above holds good so long as the central conic is referred to a pair of conjugate diameters as co-ordinate axes.

72. ASYMPTOTES. If in the equation of the tangent at any point (x_1, y_1) on a central conic we substitute the value of y_1, deduced from the fact that (x_1, y_1) is a point on the conic, we have a form which reduces at once to

$$\alpha x \pm y\sqrt{\frac{\beta}{x_1^{\,2}} - \alpha\beta} = \frac{1}{x_1}$$

The double sign before the radical arises from the fact that there are two values of y_1, and hence two tangents, corresponding to a single value of x_1. If now the value of x_1 is allowed to increase indefinitely, i. e., if the point of tangency is allowed to recede indefinitely from the origin, this equation of the tangent tends to the limiting form

$$\sqrt{\alpha}x \pm i\sqrt{\beta}y = 0$$

or $\quad bx \pm iay = 0 \quad | \quad bx \pm ay = 0$

In other words the tangents at infinity to the ellipse are a pair of imaginary lines intersecting in the center of the ellipse, and for the hyperbola a pair of real lines intersecting in the center of the hyperbola and forming the diagonals of the rectangle on the axes. Tangents at infinity to any central conic are called its asymptotes.*

PROBLEMS.

1. Find the angle between the two asymptotes of a central conic.
2. Show that the asymptotes of an equilateral hyperbola are perpendicular to each other. From this fact the equilateral hyperbola is sometimes called a rectangular hyperbola.
3. Show that any asymptote regarded as a diameter is its own conjugate.
4. Show that any two conjugate diameters of an hyperbola are separated by an asymptote.
5. Show that if the asymptotes of an hyperbola be taken as a pair of oblique axes the equation of the hyperbola reduces to the form

$$xy = \frac{a^2 + b^2}{4} \quad \text{or} \quad xy = -\frac{a^2 + b^2}{4}$$

according to the choice of positive directions of the new axes.
6. Show that the equation of the tangent at any point (x_1, y_1) on the hyperbola has the form

*This definition might be made a general definition of an asymptote to any curve were it not for the fact that in special cases the line at infinity is itself a tangent, and some mathematicians prefer to exclude it from the list of possible asymptotes.

$$\frac{x}{x_1} + \frac{y}{y_1} = 2$$

if the asymptotes are the axes of co-ordinates.*

7. Show that the segment of the tangent included between the asymptotes is bisected at the point of tangency.†

8. Show that the product of the intercepts of the tangent on the asymptotes is independent of the position of the tangent and equal to the sum of the squares of the lengths of the semi-axes.

9. Show that the area of the triangle formed by the asymptotes and any tangent is independent of the position of the tangent and equal to the product of the lengths of the semi-axes.

73. THE CONJUGATE HYPERBOLA. Closely associated with the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

there is a second hyperbola

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is turned along the Y axis in place of the X axis and has the transverse and conjugate axes of the original hyperbola as conjugate and transverse axes. It is called the conjugate of the original hyperbola and plays an interesting part in the theory of conjugate diameters.

PROBLEMS.

1. Show that if a diameter meets an hyperbola in real points its conjugate meets the conjugate hyperbola in real points and conversely.

2. Show that an hyperbola and its conjugate have the same asymptotes.

---

*The student will find it simpler to deduce the equation of the tangent directly from the equation

$$xy = \frac{a^2 + b^2}{4}$$

than to deduce it by transformation of

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

†Remember that while the formula for distance presupposes rectangularity the formulae for division of a segment in a given ratio do not.

3. Show that if a diameter meets an hyperbola in imaginary points it meets the conjugate hyperbola in real points whose co-ordinates are the moduli of the co-ordinates of the imaginary points in which it meets the original hyperbola.

4. Show that the asymptotes separate the diameters of an hyperbola into two groups (whose slopes are respectively less and greater than $\frac{b}{a}$) one meeting the original hyperbola in real points and the other meeting the conjugate hyperbola in real points.

74. THE AUXILIARY CIRCLES. The two circles having the center of the ellipse as centers and the major and minor axes of the ellipse as diameters are called the major and minor auxiliary circles. The practical importance of these circles in the theory of the ellipse is largely due to the fact that they serve to connect the ellipse with the circle, a curve for which we have a full and for the most part simple geometric treatment. The corresponding curves for the hyperbola are a pair of equilateral hyperbolas, and the lack of a corresponding geometric treatment for equilateral hyperbolas renders the analogues of the following problems of small practical importance:

PROBLEMS.

1. Show that if a point on the ellipse and one on the major auxiliary circle have equal abscissas their ordinates are in the ratio $\frac{b}{a}$.

2. Given an ellipse and its major auxiliary circle, divide the major axis into a number of equal parts and on these parts construct pairs of rectangles of which AD and AF are a sample pair. From the last problem the areas of any such pair are evidently in the ratio $\frac{b}{a}$. But the areas of the ellipse and the major auxiliary circle are evidently twice the limits of the sums of these rectangles as the number of parts into which the major axis is divided tends to infinity. Show from this that the area of the ellipse is πab.

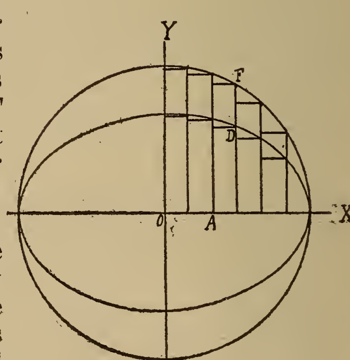

FIG. 29.

3. Find the areas of the ellipses of problems 2 and 3, article 67.

4. If through any point D on an ellipse the ordinate be drawn and extended till it meets the major auxiliary circle in the point F, then the angle FOX is called the eccentric angle of the point D. Show that if (x_1, y_1) is any point on an ellipse and θ_1 the eccentric angle of that point

$$x_1 = a \cos \theta_1 \qquad\qquad y_1 = b \sin \theta_1$$

5. In our investigation of the sub-tangent it was found that the length of the sub-tangent corresponding to the point (x_1, y_1) did not depend on either b or y_1. It follows therefore that if a family of ellipses is constructed with the same major axis but with different minor axes, an ordinate erected at any point in the major axis will cut this family of ellipses in a series of points having the same sub-tangents, i. e., the tangents at all these points meet at the same point on the major axis. This family of ellipses includes the major auxiliary circle. Hence derive a method of drawing a tangent at any point on an ellipse with ruler and compass.

CHAPTER XV.

THE PARABOLA.

75. DETERMINATION OF FORM. We take up now the consideration of type (2) of Chapter XIII, the equation

$$By^2 + 2Gx = 0,$$

in which neither B nor G is zero. Solve for y, put $-\frac{G}{B} = c$, and we have

$$y^2 = 2cx.$$

If c is positive the values of y are imaginary when x is negative, zero when x is zero, and real when x is positive. As x increases from zero to plus infinity y also increases to plus infinity for one set of values and decreases to minus infinity for the other set. The curve therefore lies wholly to the right of the Y axis, passes through the origin and extends upward and downward indefinitely. Careful plotting will show it to be of the form here given. It is called a parabola. If c is negative, all that has been said holds except that the curve lies to the left of the Y axis.

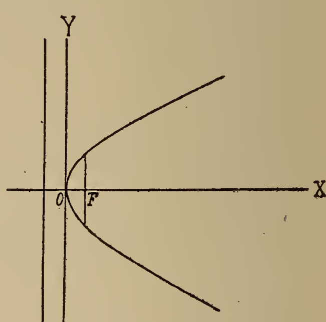

FIG. 30.

76. EARLY GEOMETRIC DEFINITION. The parabola, like the ellipse and the hyperbola, was known to geometricians before the discovery of analytic geometry. One of its geometric definitions is as follows: The parabola is the locus of a point moving in such a way that its distance from a fixed point, divided by its distance from a fixed line, is equal to unity. As in the case of the ellipse and hyperbola the fixed point is called the focus, the fixed line the

directrix and the constant ratio the eccentricity. Selecting for axes of co-ordinates the directrix and the perpendicular upon it from the focus, and denoting the distance from the focus to the directrix by d, we deduce at once from the definition just given the equation

$$y^2 = 2d\left(x - \frac{d}{2}\right).$$

The form of this equation shows at once that the X axis is an axis of symmetry, that the Y axis is parallel to the other axis of symmetry, and that the vertex of the curve is at the point $\left(\frac{d}{2}, 0\right)$. Moving the Y axis parallel to itself till it passes through this vertex we reduce the equation to the form

$$y^2 = 2dx$$

which corresponds to the form of the previous article. The definition of the curve given above shows also that the distance d equals the semi-parameter p, as defined in article 67, problem 9. We have therefore the following relations between the constants we have employed: $c = d = p$.

By means either of the geometric definition of the parabola or by consideration of the work done in problems 11, 12, 13, article 67, we see at once that the parabola is the limiting form of either the ellipse or hyperbola as the center tends to infinity under the restriction $p =$ constant, or as the eccentricity tends to unity. The student will find it both interesting and profitable to derive theorems for the parabola by an examination of the limiting form of the corresponding theorems for the ellipse or hyperbola.

77. MECHANICAL CONSTRUCTION. To trace the parabola mechanically we place one edge of a rectangular board CE against the directrix AB, fasten one end of a cord equal in length to CD at D and the other end at the focus F, and slide the board along the directrix, keeping the cord pressed against the edge CD by a pencil point P. The point P will trace the parabola, for in every position we have $PF = PC$.

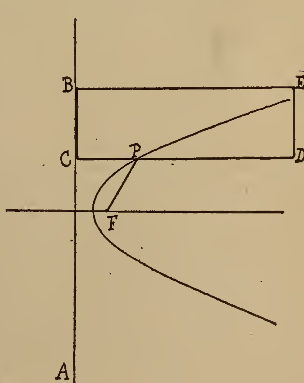

FIG. 31.

PROBLEMS.

1. Show that the vertex of a parabola is midway between the focus and the directrix.

2. Find the focal radius of any point (x_1, y_1) on the parabola

$$y^2 = 2px$$

3. Find the foci, parameters, and directrices of the following parabolas.

$y^2 = 10x$	$y^2 = -4x$
$y^2 = ax + b$	$y^2 = -4x + 5$
$x^2 = 4y$	$x^2 = 3y + 1$
$y^2 - x + 1 = 0$	$x^2 + 2y - 1 = 0$
$y^2 = 3x + 4$	$3y^2 + 4x - 2 = 0$

4. Write the equations of the following parabolas:
   - $x - 2 = 0$ is the directrix, and (3, 4) the focus.
   - (3, 4) is the focus, axis of symmetry is parallel to the Y axis, curve opens upward, parameter is 6.
   - (2, 3) is the vertex, (1, 3) the focus.

78. DIAMETERS. The equation of the diameter of the parabola may be regarded as a special case of the general equation developed in article 62, or it may be found directly by determining the locus of the middle points of a system of parallel chords. If the latter method is adopted, the student should remember that in any locus problem the thing sought is the equation which determines the movement of the tracing point. If then, as in the present instance, there is developed in the course of the discussion an equation of the type

$$y = k$$

(where k is a constant and y is a co-ordinate of the tracing point) this equation in itself determines the path of the tracing point, and is therefore the equation desired.

PROBLEMS.

1. Write the equation of the diameter of the parabola $y^2 = 2px$ bisecting the chords of slope m.

2. Given a parabola $y^2 = 2px$, and a diameter $y = 4a$, what is the slope of the chords which it bisects?

3. Show that all diameters of a parabola are parallel to the axis of symmetry; first from the form of the equation of the diameter, and then from the location of the center.

4. Why do we not develop for the parabola the theory of conjugate diameters and supplemental chords?

79.
TANGENTS AND NORMALS.

PROBLEMS.

1. What condition must be satisfied by the coefficients of the line

$$y = mx + h$$

in order that it may be tangent to the parabola

$$y^2 = 2px?$$

2. Develop the equation whose roots are the slopes of the tangents from the point (x_1, y_1) to the parabola

$$y^2 = 2px$$

Hence show that if the point is not on the conic two tangents to the conic can be drawn through it.

3. Show that the equation of the tangent to the parabola

$$y^2 = 2px$$

at the point (x_1, y_1) is

$$yy_1 = p(x + x_1)$$

4. Find the equation of the normal at the same point.

5. Find the length of the sub-tangent and show that it is bisected at the vertex.

6. Show that the subnormal is constant and equal to the semi-parameter.

7. Hence deduce geometric constructions for the tangent and normal at any point of a parabola whose focus and axis are given.

8. Show that the two tangents which may be drawn from any point to a parabola are real or imaginary according as the point is without or within the parabola.

9. Show that the tangent and normal at any point bisect the angles formed by the focal radius of the point and the diameter through the point. Show also that this is a special case of problem 9, article 70.

10. Hence deduce geometric constructions for the tangent and normal at any point on a parabola whose focus and axis are given.

11. Note that problem 9 is equivalent to the statement that the tangent makes equal angles with the axis of symmetry and the focal radius through the point of tangency, and hence deduce a geometric construction for the tangent and normal.

12. Find the locus of the points from which two perpendicular tangents may be drawn to a parabola.

13. Find the locus of the foot of the perpendicular let fall from the focus upon a variable tangent to the parabola, and show that it is degenerate and consists of the tangent at the vertex and a pair of imaginary lines.

14. Hence deduce a geometric construction for the tangent at any point of a parabola whose focus and axis are given.

15. Show that problems 12 and 13 are special cases of the results obtained in problems 11 and 12 of paragraph 70.

16. Show that the perpendicular distance from the focus to any tangent is equal to $\frac{1}{2}\sqrt{2pr}$, where r is the focal radius of the point of tangency.

17. Show that the tangent at the extremity of any diameter is parallel to the chords bisected by that diameter.

18. Deduce the equation of the parabola referred to any diameter and the tangent at the extremity of that diameter as oblique axes, and show that it reduces to the form

$$y^2 = 2p'x,$$

where p' is a new constant. What is the value of p' in terms of p and the angle between the new axes?

19. The slope of the tangent to the parabola $y^2 = 2px$ at the point (x_1, y_1) is $\frac{p}{y_1}$. Since (x_1, y_1) is on the conic this may be reduced to $\sqrt{\frac{p}{2x_1}}$. Show that when the point of tangency removes toward infinity the tangent tends to parallelism with the axis of symmetry of the parabola.

80. POLES AND POLARS. Build up the theory of poles and polars for the parabola.

CHAPTER XVI.

ADDITIONAL WORK ON THE GENERAL EQUATION OF THE SECOND DEGREE.

81. NECESSITY OF A GENERAL TREATMENT. In Chapter XIII we investigated such properties of the curve represented by the general equation of the second degree as would guide us in reducing the equation to certain standard or type forms. By the aid of these type forms any investigation which deals with the curve as an individual and without regard to its relative position may evidently be carried on. For example, the theorem that the locus of the points of intersection of perpendicular tangents to an ellipse is a circle concentric with the ellipse is altogether independent of the position of the ellipse, and therefore is proved with entire generality by the use of the type form. On the other hand, any theorem which depends upon the position of a conic with respect to the axes or to other curves cannot be established in its most general form by the use of type forms, since these presuppose a special position of the curve with respect to the axes. For example, the condition of tangency, equation of polar, values of sub-tangent and sub-normal hitherto deduced are all based upon the assumption that the curve occupies a particular position, and therefore they cannot be applied to the same curve in any other position. So long as we are dealing with a single curve there is no reason why we should not assume a particular position for it, but when two or more curves enter into the discussion only one of them in general can be given the desired position, and it is therefore necessary to make an investigation of the conic without regard to its position, i. e., to make a study of the general equation of the second degree.

The results already secured are contained in a series of theorems in the opening paragraphs of Chapter XIII.

82. DEGENERATE CONICS. When the left hand member of an equation consists of two or more rational factors, the equation is satisfied by any set of values of the variables which make any one of the factors equal to zero. In consequence the locus corresponding to such an equation consists of two or more parts, which are themselves the loci representing the equations formed by equating the various factors to zero. Such a locus is said to be degenerate. When a conic degenerates it is evident that the factors of the left hand member must be of the first degree and hence that the conic must degenerate into a pair of straight lines. The only questions of interest in such a case are concerning the point of intersection and the angle between the lines. The point of intersection evidently meets the definition of the center of symmetry and consequently may be found by solving the equations

$$ax + hy + g = 0$$
$$hx + by + f = 0$$

To find the angle between the lines assume that the conic degenerates into the two lines

$$\alpha_1 x + \beta_1 y + \gamma_1 = 0$$
$$\alpha_2 x + \beta_2 y + \gamma_2 = 0.$$

Then the equation of the conic is

$$\alpha_1\alpha_2 x^2 + \beta_1\beta_2 y^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)xy + (\alpha_1\gamma_2 + \alpha_2\gamma_1)x + (\beta_1\gamma_2 + \beta_2\gamma_1)y + \gamma_1\gamma_2 = 0$$

Consider the two lines parallel to these lines and passing through the origin

$$\alpha_1 x + \beta_1 y = 0$$
$$\alpha_2 x + \beta_2 y = 0$$

These taken together constitute a degenerate conic

$$\alpha_1\alpha_2 x^2 + \beta_1\beta_2 y^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)xy = 0$$

We may at once state the theorem that the second degree part of the equation of a degenerate conic is the left hand member of the equation of a second degenerate conic whose lines intersect in the origin and are parallel to the lines of the first conic.

It follows that if

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

represents a degenerate conic, the angle between the lines is equal to the angle between the lines of

$$ax^2 + by^2 + 2hxy = 0$$

This equation may be at once factored and the angle θ between the lines expressed by the relation

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

83. DISCRIMINANT. It is desirable to have some test to apply to a second degree equation in order to determine at once whether or not it is degenerate. A moment's consideration will show that one peculiarity of a degenerate conic is that the center of symmetry, i. e., the point of intersection of the two component lines, is on the conic; and, conversely, the center of symmetry can be on the conic only when the conic degenerates. The necessary and sufficient condition for degeneracy of a conic is therefore the existence of a point which satisfies the three equations

$$\mathrm{I}\begin{cases} ax + hy + g = 0 \\ hx + by + f = 0 \\ ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0. \end{cases}$$

The last of these may be written in the form

$$x(ax + hy + g) + y(hx + by + f) + gx + fy + c = 0$$

Equations I are therefore equivalent to

$$\begin{aligned} ax + hy + g &= 0 \\ hx + by + f &= 0 \\ gx + fy + c &= 0. \end{aligned}$$

But the necessary and sufficient condition for the simultaneous satisfaction of these three equations is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

This determinant, which plays an important part in the theory of conics, is called the discriminant of the equation of the conic. We shall denote it by Δ.

84. CLASSIFICATION. After it has been shown that a conic is not degenerate, the most important question is whether it is an ellipse, hyperbola, or parabola. Article 72 and Problem 19, article 79, have shown that the ellipse has only imaginary points at infinity, that the hyperbola extends to infinity in two different directions, and that the two sides of the parabola tend to parallelism as the tracing point goes off to infinity. If then we join the infinite points on a conic to any point in the plane the resulting pair of lines will have imaginary slopes for the ellipse, real and equal slopes for the parabola, real and unequal slopes for the hyperbola. The lines which run from any point to the infinite points on the conic are the lines which give infinite values for r in the equation of article 59, i. e., the lines whose slopes are such that we have

$$al^2 + 2hlm + bm^2 = 0.$$

The factors of this are imaginary, real and equal, or real and unequal according as $ab - h^2$ is positive, zero, or negative. An equation of the second degree in two variables therefore represents an ellipse, parabola, or hyperbola according as $ab - h^2$ is positive, zero, or negative.

85. TANGENTS AND NORMALS. Since the tangent is a line meeting the conic in two coincident points, the condition that a line through (x_1, y_1) may be tangent to the conic represented by the general equation of the second degree is that the two values of r given by the equation

$$f(x_1, y_1) + 2r[l(ax_1 + hy_1 + g) + m(hx_1 + by_1 + f)] \\ + r^2(al^2 + 2hlm + bm^2) = 0^*$$

shall be equal. The condition for equal roots gives

$$\text{I} \qquad f(x_1, y_1)(al^2 + 2hlm + bm^2) \\ -[l(ax_1 + hy_1 + g) + m(hx_1 + by_1 + f)]^2 = 0,$$

an equation of the second degree in the ratio $\frac{m}{l}$ whose two roots are the slopes of the two tangents from (x_1, y_1) to the conic.

$^*f(x_1, y_1) \equiv ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c.$

Eliminating $\frac{m}{l}$ between this equation and the equation of the line through (x_1, y_1)

II $$\frac{x - x_1}{l} = \frac{y - y_1}{m}$$

we have

III $$f(x_1, y_1)\,[a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2] \\ -[(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f)]^2 = 0$$

as the equation of the pair of tangents from (x_1, y_1) to the conic.*

If (x_1, y_1) is on the curve, $f(x_1, y_1) = 0$, the two tangents coincide, the equation of the pair of tangents reduces to a perfect square, and we have as the equation of the tangent at the point (x_1, y_1) on the conic

$$(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0.$$

By transposing the negative terms to the right hand member and adding

$$gx_1 + fy_1 + c$$

to both sides, this last equation is reduced to the form

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = \\ x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + gx_1 + fy_1 + c \equiv \\ f(x_1, y_1) = 0.$$

we have therefore for the equation of the tangent at the point (x_1, y_1) on the curve

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0,$$

which may also be written in the form

$$x_1(ax + hy + g) + y_1(hx + by + f) + gx + fy + c = 0,$$

as the student will see on multiplying out the two forms.

The normal at any point on the conic is the perpendicular to the tangent at that point. With this definition the student should have no difficulty in writing its equation.

---

*The significance of this elimination may not seem clear to the student. Equation II states that the point (x, y) is on a line through (x_1, y_1) with direction cosines l, m. Equation I states that l and m are so determined that the line is tangent. Equation III, deduced by making I and II simultaneous, states that (x, y) is on one or the other of the tangent lines.

86. A SECOND CONDITION OF TANGENCY.

The condition of tangency given in the last article is applicable when we have a line through a fixed point. We frequently need the condition of tangency applicable when the line is given by a general equation of the first degree. Let

$$\alpha x + \beta y + 1 = 0$$

be a given straight line, what is the condition which must be satisfied in order that it may be tangent to the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0?$$

Let us assume the line to be tangent at a point (x_1, y_1). Its equation must then be

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0.$$

This equation represents the same line as

$$\alpha x + \beta y + 1 = 0$$

Therefore

$$\frac{ax_1 + hy_1 + g}{\alpha} = \frac{hx_1 + by_1 + f}{\beta} = \frac{gx_1 + fy_1 + c}{1}$$

For convenience of elimination equate each of these ratios to $-\mu$ and we have

$$\begin{aligned} ax_1 + hy_1 + g + \quad &= 0 \\ hx_1 + by_1 + f + \beta\mu &= 0 \\ gx_1 + fy_1 + c + \mu &= 0 \\ \alpha x_1 + \beta y_1 + 1 &= 0 \end{aligned}$$

the last of which holds true because the point (x_1, y_1) is on the line

$$\alpha x + \beta y + 1 = 0.$$

The necessary and sufficient condition for the co-existence of these four equations is

$$\begin{vmatrix} a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & 1 \\ \alpha & \beta & 1 & 0 \end{vmatrix} = 0$$

which is therefore the condition which must be satisfied by the coefficients of the given line and conic in order that they may be tangent to each other.

Expand this determinant, arrange the terms according to powers of α and β, denote the minors of the determinant, taken with their proper signs, by $A, B, C, F, G, H,$ as usual, and the condition of tangency takes the form

$$A\alpha^2 + B\beta^2 + 2H\alpha\beta + 2G\alpha + 2F\beta + C = 0\,*$$

a form which we shall hereafter denote by

$$F(\alpha, \beta) = 0$$

From this general condition of tangency the student may at once deduce as special cases the conditions developed in articles 49, 70, and 79.

What is the vital difference between the equation just developed and the equation of the conic? $f(x, y) = 0$ is a condition which must be satisfied by the co-ordinates x and y of all points on the conic, i. e., the condition which selects from all the points in the plane those that lie on the conic. Similarly, $F(\alpha, \beta) = 0$ is the condition which must be satisfied by the co-efficients of every line tangent to the conic, i. e., the condition which selects from all the lines in the plane those which are tangent to the conic. But a conic is just as fully determined by the aggregate of its tangents as by the aggregate of its points. The following questions immediately present themselves. (a) Since the co-efficients α and β determine the position of the line, are they not in some sense co-ordinates of the line? (b) Is not $F(\alpha, \beta) = 0$ just as truly the equation of the conic as $f(x, y) = 0$? (c) Is it not possible to build up a geometry in which the variable element is the line rather than the point? (d) If so, would not the algebraic work of the two geometries be identical; and, therefore, could we not infer from each theorem already developed a new one differing from the old by an interchange of point and line? The answers to these questions lie beyond the scope of the present volume, but the student who will follow them up will find that they lead into one of the most interesting fields of modern mathematical investigation.

*An immediate expansion of the determinant in this form is possible. Consult in any standard text on determinants the theorem on the expansion of a determinant in terms of the products in pairs of the constituents of any row and column.

87. POLE AND POLAR. Any one of the methods used for determining the equation of the polar of a point with respect to a conic might be extended to the general conic. We consider the one which regards the polar as the locus of harmonic conjugates of the pole with respect to the intersections of the conic and the chords through the pole. (This method was developed for the circle in article 55, and the notation and diagram of that article will serve for this also.) Let

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

be any conic, and (x_1, y_1) any point P. Write the equations of any line through (x_1, y_1) in the form

$$x = x_1 + lr$$
$$y = y_1 + mr.$$

Then the distances from the point (x_1, y_1) to the conic, measured along this line, are given by the equation

$$f(x_1, y_1) + 2r[l(ax_1 + hy_1 + g) + m(hx_1 + by_1 + f)] + r^2(al^2 + bm^2 + 2hlm) = 0$$

Denote these distances by r_1 and r_2. Then

$$\frac{1}{r_1} + \frac{1}{r_2} = -\frac{2l(ax_1 + hy_1 + g) + 2m(hx_1 + by_1 + f)}{f(x_1, y_1)}$$

Let the harmonic conjugate, S, of (x_1, y_1) with respect to the two intersections of the chord and the conic be denoted by (x', y'), then

$$\frac{2}{PS} = \frac{2}{\sqrt{(x' - x_1)^2 + (y' - y_1)^2}}$$

We have also

$$l = \frac{x' - x_1}{\sqrt{(x' - x_1)^2 + (y' - y_1)^2}} \qquad m = \frac{y' - y_1}{\sqrt{(x' - x_1)^2 + (y' - y_1)^2}}$$

The necessary and sufficient condition that S may be the harmonic conjugate of P with respect to the two intersections of the chord and the conic is

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{PS}.$$

Substitute the values deduced above and reduce and we have, on dropping accents,

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0$$

which may also be written in the form

$$x_1(ax + hy + g) + y_1(hx + by + f) + gx + fy + c = 0.$$

Either of these is therefore the equation of the polar of (x_1, y_1) with respect to the conic $f(x, y) = 0$.

The majority of the properties already established for poles and polars do not depend at all upon the choice of the system of reference, and therefore the demonstrations already given hold also for the case of the general conic. If the student is not satisfied as to the generality of any particular theorem, he should investigate the subject by the aid of the general equation just developed.

88. LENGTH OF AXES. Articles 61 to 63 enable us to find the equations of the axes of symmetry of any conic. This done, in any particular case it is theoretically an easy matter to find the distances between the intersections of these axes with the conic, i. e., the lengths of the axes. Practical difficulties of computation are apt to arise on account of the frequent presence of irrationals in the equations. These difficulties may be minimized by replacing the irrational by its decimal value carried to such a degree of approximation as the particular investigation may demand. It is also well to remember that what is needed is not the co-ordinates x_1, y_1, x_2, y_2 of the intersections, but the quantities $(x_1 - x_2)^2$ and $(y_1 - y_2)^2$; and that if α and β are the roots of

$$ax^2 + 2bx + c = 0,$$

$$(\alpha - \beta)^2 = \frac{4b^2 - 4ac}{a^2}. *$$

89. FOCI AND ECCENTRICITY. When the location of the center and the lengths and directions of the axes of a conic have been determined, the location of the foci and the directrices and the determination of the eccentricity immediately follow.†

90. ASYMPTOTES. It has already been shown that the asymptotes of any central conic

$$\alpha x^2 + \beta y^2 = 1$$

*An interesting treatment of this problem is given in C. Smith's Conic Sections, Article 171.

†For a proof of the existence of the four foci different from the one given in Article 66, see C. Smith, Conic Sections, Article 190.

referred to its axes of symmetry as axes of co-ordinates, are given by the two factors of

$$\alpha x^2 + \beta y^2 = 0$$

If now we apply any transformation or series of transformations, of the sort hitherto considered, and thus transform the equation of the conic to the form

$$\text{I} \qquad ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

the equation of the pair of asymptotes will become

$$\text{II} \qquad ax^2 + by^2 + 2hxy + 2gx + 2fy + k = 0,$$

i. e., the equation of the pair of asymptotes of any central conic differs from the equation of the conic only in the constant term. (At first glance it seems as if c and k are connected by the relation, $c = k - 1$, but a moment's consideration will show that constant factors may have been introduced at any point in the transformation and that in consequence c and k may differ by any constant.) The determination of k presents, however, no serious difficulty. Equation II is an equation with one arbitrary parameter, k, and therefore represents a family of conics distinguished from each other by the various values of k. In this family the pair of asymptotes is included, and the value of k corresponding to this particular member of the family may be determined by giving algebraic expression to any one of its additional properties, just as in article 43 we determined a particular member of a family of lines by giving algebraic expression to some property of the line other than the one which it shared with all the members of the family. Now the conic whose equation we are seeking, the pair of asymptotes, is degenerate and this property of degeneracy is certainly not shared by all the conics of the family. If therefore we impose on the equation II the condition of degeneracy, we shall get the values of k corresponding to the degenerate members of the family; included among these will be the value of k corresponding to the pair of asymptotes. But the condition of degeneracy is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & k \end{vmatrix} = 0$$

an equation of the first degree in k. There is therefore only one

degenerate member of the family;* and the value of k thus determined gives us, when substituted in equation II, the equation of the pair of asymptotes of the conic represented by equation I.

91. SPECIAL TREATMENT FOR THE PARABOLA. If the conic represented by our general equation of the second degree is in any particular case a parabola, certain steps in our general treatment (such as the determination of the center and the lengths of the axes) become impracticable, since they introduce infinite quantities into the discussion. What we actually need in order to determine the nature of a parabola from its equation is the location of its axis of symmetry and the tangent at its vertex, and the value of its parameter. The discussion to follow makes use of the following theorems:

(A) The semi-parameter of a parabola is equal to the distance of any diameter from the axis of symmetry of the parabola, multiplied by the tangent of the angle which the chords bisected by that diameter make with the axis of symmetry. (This theorem is a mere generalization of the solution of problem 1, article 78.)

(B) Let

$$S_1 = 0 \qquad \text{and} \qquad S_2 = 0$$

be any two straight lines. Then

$$S_1^2 + kS_2 = 0$$

is a conic passing through the intersection of these two lines, with $S_2 = 0$ as its tangent at this intersection. For if we substitute the value of y derived by solving $S_2 = 0$ in the equation $S_1^2 + kS_2 = 0$ the resulting equation in x is a perfect square, i. e., the line $S_2 = 0$ meets the conic $S_1^2 + kS_2 = 0$ in two coincident points (is tangent), and since, as is easily seen, the co-ordinates of these points satisfy both $S_1 = 0$ and $S_2 = 0$ the point of tangency is the intersection of the two lines.

Let it be granted that the general equation

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

represents a parabola. Then we have

$$ab - h^2 = 0,$$

*That is, only one with a finite value of k. But the asymptotes of a central conic are a pair of lines in the finite part of the plane and hence all the co-efficients in their equation are finite.

and the general equation of a diameter takes the form

$$ax + \sqrt{ab}\, y + g + \frac{m}{l}(\sqrt{ab}\, x + by + f) = 0$$

that is $$(a + \frac{m}{l}\sqrt{ab})\, x + (\sqrt{ab} + \frac{m}{l} b)\, y + g + \frac{m}{l} f = 0$$

(a result which gives us the theorem that all diameters of a parabola are parallel). For the axes of symmetry we must have

$$\left(\frac{m}{l}\right)^2 + \frac{a - b}{\sqrt{ab}}\left(\frac{m}{l}\right) - 1 = 0$$

i. e., $$\frac{m}{l} = -\sqrt{\frac{a}{b}} \text{ or } \sqrt{\frac{b}{a}}$$

Substituting the first of these values in the general equation of a diameter we have a constant equal to zero, i. e., one of the diameters of the parabola is the line at infinity, a fact which we already knew. Substituting the other value we have

$$(a + b)\sqrt{a}x + (a + b)\sqrt{b}y + \sqrt{a}g + \sqrt{b}f = 0$$

i. e., $$\sqrt{a}x + \sqrt{b}y + \frac{\sqrt{a}g + \sqrt{b}f}{a + b} = 0$$

the equation of the axis of symmetry of the parabola. Since all diameters of the parabola are parallel, the particular one which passes through the origin is

$$\sqrt{a}x + \sqrt{b}y = 0$$

and the perpendicular distance from this diameter to the axis of symmetry is

$$\frac{\sqrt{a}g + \sqrt{b}f}{(a + b)^{\frac{3}{2}}}$$

The original equation may be put in the form

$$(\sqrt{a}x + \sqrt{b}y)^2 + 2gx + 2fy + c = 0$$

Therefore $$2gx + 2fy + c = 0$$

is the tangent to the parabola at the point where it is met by the diameter through the origin. The slope of this tangent and therefore of the chords bisected by the diameter

$$\sqrt{a}x + \sqrt{b}y = 0$$

is $-\frac{g}{f}$ and the tangent of the angle which these chords make

with the axis of symmetry is

$$\frac{\sqrt{af}-\sqrt{bg}}{\sqrt{ag}+\sqrt{bf}}\cdot$$

therefore
$$p=\frac{\sqrt{af}-\sqrt{bg}}{(a+b)^{\frac{3}{2}}}.$$

The student has now sufficient material at hand to determine all the data concerning any particular parabola.

Example. As an illustration consider the equation

$$4x^2+4xy+y^2+6x+2y+4=0,$$

i. e.,
$$(2x+y)^2+6x+2y+4=0.$$

The axis of symmetry is

$$2x+y+\frac{7}{5}=0$$

and the semi-parameter is $\pm\frac{1}{25}\sqrt{5}$

The vertex of the conic is its intersection with the axis of symmetry, i. e.,

$$\left(-\frac{79}{50},\ \frac{44}{25}\right)$$

The tangent at the vertex is therefore

$$\left(y-\frac{44}{25}\right)=\frac{1}{2}\left(x+\frac{79}{50}\right)$$

and the directrix is parallel to this at a distance of $\frac{1}{50}\sqrt{5}.$

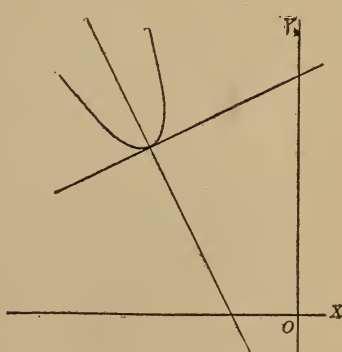

FIG. 32.

All that is needed to complete our information is to know which way the parabola is turned. This is settled at once by the co-ordinates of any other point on the conic. For example the diameter

$$2x+y=0$$

meets the curve at $(-2,\ 4)$ which lies to the left of the tangent at the vertex. The curve is therefore as drawn.

92. HIGHER LOCI. The student has now at his disposal sufficient formulae to enable him to determine the character, form, and location of the curve represented by any algebraic equation of the second degree. Loci corresponding to higher degree algebraic equations or to transcendental equations, as well as curves and surfaces in space, can be treated in a more satisfactory way after the student has acquired a knowledge of the elements of the differential and integral calculus.

93. FAMILIES OF CONICS.

PROBLEMS.

1. Given that

$$S_1 = 0 \qquad \text{and} \qquad S_2 = 0$$

are the equations of two conics, show that

$$S_1 + kS_2 = 0,$$

where k is an arbitrary constant, represents a family of conics, singly infinite in number, each of which passes through all four of the intersections of the two original conics.

2. Five points on a conic are sufficient to determine it uniquely. Therefore among the conics through four given points, one and only one passes through each of the remaining points of the plane. Hence show that the family

$$S_1 + kS_2 = 0$$

includes every possible conic through the intersections of

$$S_1 = 0 \qquad \text{and} \qquad S_2 = 0.$$

3. Given the two conics

$$3x^2 + 2y^2 + 4x - 1 = 0$$
$$x^2 - y^2 + 2y - 3 = 0$$

form the equation of the family of conics through the intersections of these two conics, and find the equation of that member of the family which passes through the origin.

4. Show that the family of the last problem includes one circle, two parabolas, and three degenerate members. To what extent are these statements true of the general case of problem 1?

5. How many members of the family in problem 1 are tangent to any given line?

CHAPTER XVII.

OTHER SYSTEMS OF CO-ORDINATES, POLAR CO-ORDINATES.

94. VARIOUS SYSTEMS OF CO-ORDINATES. We must not assume that the system of co-ordinates we have been using is the only system in use. On the contrary any set of quantities which will serve to determine the position of a point may be taken as co-ordinates of the point.

In the Cartesian system the co-ordinates of a point are the distances of two lines, $x=a$, $y=b$, from the base lines which form the system of reference. In other words the point is located as the intersection of two lines.

Next in point of simplicity comes a system in which the point is located as the intersection of a straight line and a circle. In this the system of reference is a fixed point A and a fixed line AB. Any point P in the plane may now be located by giving ρ the radius of the circle centered at A and passing through P, and θ the angle between the base line AB and the line AP. It is evident that in this system a pair of co-ordinates, ρ, θ, determines the point P not uniquely, but as one of two, i. e., either P or P'.

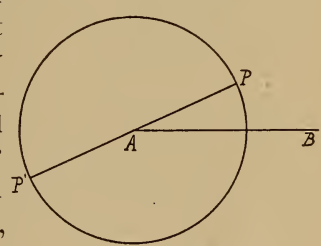

FIG. 33.

Next in order of simplicity comes a system of bi-polar co-ordinates in which the point is located as the intersection of two circles. The system of reference consists of two fixed points A and B and the two co-ordinates are the radii of two circles centered at A and B. It is evident that in this system a pair of co-ordinates does not determine a point uniquely, but as one of two, i. e., either P or P'.

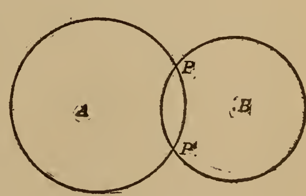

FIG. 34.

Consider still another system. Take two ellipses, concentric and co-axial, and let the semi-axes of the one be a and $2a$, and the semi-axes of the other $2b$ and b. Then a and b are the co-ordinates of the intersections of the two ellipses. It is evident that any pair of co-ordinates does not determine a point uniquely, but as one of four, P, P', P'', P'''.

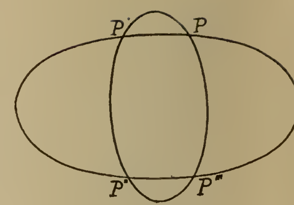

FIG. 35.

In general given any two families of curves, each of which depends upon a single arbitrary parameter, and given also a point P, we can determine the values of the parameters giving in each family the particular member which passes through P. This pair of values may be regarded as the co-ordinates of P, determining P, not uniquely, but as one of the intersections of the two curves given by the chosen values of the parameters. It is now evident that the number of systems of co-ordinates is infinite.

95. MERITS AND DEMERITS OF VARIOUS SYSTEMS. Certain systems of co-ordinates possess particular merit for the investigation of some particular problem. In the bi-polar for example, if we take the base points as the foci of an ellipse or hyperbola the equations of these curves reduce to the simple forms $x + y = k$, and $x - y = k$. On the other hand each system has certain demerits. In the bi-polar system for example, the simplicity of the form taken by the equations of certain conics is more than offset by the complexity of the equation of the straight line.

In one particular all of the systems so far discussed are seriously lacking. Given two algebraic variables of the most general type, $x = x_1 + ix_2$, and $y = y_1 + iy_2$, we can assign to x_1, x_2, y_1, y_2 any values whatever, and therefore can form a quadruply infinite number of pairs of values of x and y. But all these systems of co-ordinates attempt to represent pairs of values of x and y by points in the plane, and such points are only doubly infinite in number. In consequence each system must fail to give a complete geometric representation to the algebraic relations under consideration. Some of the systems, as may be seen from the problems which follow this article, leave pairs of real values of the

variables without point representation, while others represent real points by pairs of imaginary values of the variables.

The majority of the systems of co-ordinates which may be used fail also in another important particular in that they do not establish a one to one correspondence between points and pairs of values, but determine two or more points as corresponding to a single pair of values. One of the great beauties of the Cartesian system is that it establishes a one to one correspondence between the points of the plane and all pairs of real values of two algebraic variables.

PROBLEMS.

1. Given a system of bi-polar co-ordinates in which the distance between the base points is 8, plot the locus $x - y = 0$.
2. Is the point (3, 3) on the above locus? locate it in the diagram.
3. Plot the ellipse $x + y = 10$ in the same system.
4. Plot the hyperbola $x - y = 10$ in the same system.
5. What conditions must be met by the co-ordinates of a point in order that it may be represented on the diagram if the distance between the base points is k?
6. Given the two families of ellipses whose equations in Cartesian co-ordinates are

$$x^2 + \frac{y^2}{a^2} = 1, \quad \text{and} \quad \frac{x^2}{b^2} + y^2 = 1$$

then the values of a and b determining the ellipses through any point are the co-ordinates of that point in the new system in which the ellipses are the determining elements. What are the Cartesian co-ordinates of the points whose co-ordinates in the new system are (4, 2), (4, 1), (1, 1), $(\frac{1}{2}, 2)$?

7. Plot in this same system the locus for which $a = b$.
8. What are the co-ordinates in this system of the point whose Cartesian co-ordinates are (3, 3)?

96. POLAR CO-ORDINATES. Polar co-ordinates are of particular value in any investigation in which the important elements are the distance and direction of the variable point from a fixed point. The movement of the earth about the sun or any problem concerning a spiral curve are illustrations.

The drawbacks to the system as it was outlined in article 94 are numerous. Given a pair of values ρ, θ, we note first that while θ may have any value ρ must be positive and that the point is determined only as one of two. On the other hand given a point P we have for it a single value of ρ, but any number of values of θ i.e., $\theta, \theta+2\pi, \theta+4\pi, \theta+6\pi \ldots \theta+2n\pi$. (Hereafter we shall call the ρ of any point the radius vector and the θ the amplitude of the point.) In other words any real amplitude may be paired with any real positive radius vector and the combination will be represented by either one of two points, while a point is represented by a real radius vector paired with any one of an infinity of amplitudes differing from each other by integer multiples of 2π.

Mathematicians are accustomed however to make an assumption which enables them to include negative values of the radius vector. Abandoning the circle they fix their attention on the angle θ and the distance ρ, and agree that the positive value of ρ is to be measured from the vertex of the angle in the direction of the boundary of θ and a negative value of ρ in the opposite direction. With this agreement the co-ordinates (ρ, θ) denote the point P and the co-ordinates $(-\rho, \theta)$ the point P'. But the point P has now the co-ordinates (ρ, θ) or $(-\rho, \theta+\pi)$, or more generally $(\rho, \theta+2n\pi)$ or $(-\rho, \theta+(2n+1)\pi)$. In other words any real pair of values of ρ and θ is represented by a single point while any point has two radii vectores and an infinity of amplitudes differing from each other by integer multiplies of π.

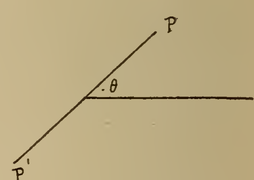

FIG. 36.

PROBLEMS.

Plot the curves corresponding to the following equations:

1. $\rho = 4$.
2. $\theta = 2$.
3. $\rho = \theta$.
4. $\rho = \sin\theta$.
5. $\rho = e^{\theta}$.
6. $\theta^2 + 3\theta + 2 = 0$.
7. Show that the area of a triangle with one vertex at the pole (the fixed point) and the others at (ρ_1, θ_1) and (ρ_2, θ_2) is $\frac{1}{2}\rho_1\rho_2 \sin(\theta_1 - \theta_2)$.
8. Show that the area of the triangle whose vertices are (ρ_1, θ_1), (ρ_2, θ_2), (ρ_3, θ_3) is $\frac{1}{2}\left\{\rho_1\rho_2\sin(\theta_1-\theta_2) + \rho_2\rho_3\sin(\theta_2-\theta_3) + \rho_3\rho_1\sin(\theta_3-\theta_1)\right\}$

97. THE RELATION OF POLAR AND CARTESIAN CO-ORDINATES. Cartesian and polar co-ordinates are connected by simple relations which make the transformation from one to the other an easy matter. We consider first the case where the pole is at the origin and the base line coincides with the X axis. In this case we have at once

$$x = \rho \cos \theta$$
$$y = \rho \sin \theta$$

whence

$$\rho = \sqrt{x^2 + y^2},$$
$$\theta = \tan^{-1} \frac{y}{x}.$$

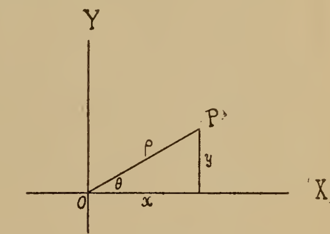

FIG. 37.

When the pole is at the origin and the base line OA makes an angle a with the axis of X we have

$$x = \rho \cos (\theta + a)$$
$$y = \rho \sin (\theta + a)$$

whence

$$\rho = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1} \frac{y}{x} - a$$

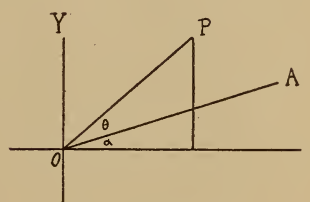

FIG. 38.

When the Cartesian system is not in either one of these positions it can easily be placed in one or the other by a movement of the axes parallel to themselves. Therefore in the most general case when the pole is at the point (a, b) and the base line $O'A$ makes an angle a with the axis of X we have

$$x - a = \rho \cos (\theta + a)$$
$$y - b = \rho \sin (\theta + a)$$

whence

$$\rho = \sqrt{(x - a)^2 + (y - b)^2}$$
$$\theta = \tan^{-1} \frac{y - b}{x - a} - a$$

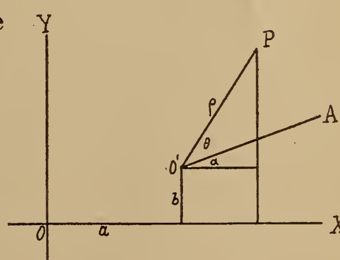

FIG. 39.

If it is desired to pass from one polar system to another whose pole is not coincident with the first it is simpler to transform first to a

Cartesian system whose origin and X axis coincide with the pole and base line of the first system and then transform to the second polar system.

PROBLEMS.

1. Show that the equation of the circle centered at (ρ_1, θ_1) and of radius r is $\rho^2 - 2\rho\rho_1 \cos(\theta - \theta_1) + \rho_1^2 - r^2 = 0$.

2. Find the equation of a circle when the pole is on the circumference and the base line is the tangent at the pole.

3. Find the equations of the various conic sections when the pole is at one focus and the base line is the axis of symmetry through that focus, and show that any one of them may be reduced to the form

$$\frac{p}{\rho} = 1 - e \cos\theta$$

where p and e are the semi-parameter and the eccentricity.

4. Let AB be the directrix, F the focus, and P any point on the conic. Deduce the equation just given directly from the diagram. (It is sometimes convenient to measure the angle θ from FC as a base line. In that case the equation takes the form

$$\frac{p}{\rho} = 1 + e \cos\theta).$$

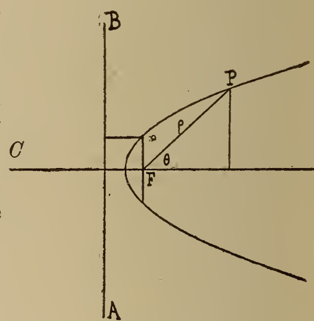

FIG. 40.

APPENDIX A.

INFINITIES OF VARIOUS ORDERS.

The student must bear in mind the fact that he is using the word infinity in a technical sense differing somewhat from the ordinary literary and philosophical usage of the word. In its ordinary usage infinity denotes that which exceeds all limitations and therefore no comparisons between various infinities are possible. In mathematical usage infinity denotes that which increases indefinitely, and it is evident that between two such quantities a perfectly definite comparison may be made. Consider some illustrations for the sake of clearness. In each of the three expressions

$$\frac{2x+1}{x} \qquad \frac{x^2+1}{x} \qquad \frac{x^2+1}{x^3}$$

let x increase indefinitely. As this happens both numerator and denominator of each fraction increase indefinitely, i. e., in technical phrase, become infinite, but the three fractions respectively tend to 2, increase indefinitely (tend to infinity), and tend to zero. In other words, in the first case while both numerator and denominator increase they remain easily comparable with each other; in the second case the numerator becomes incomparably greater than the denominator, and in the third case the numerator becomes incomparably less than the denominator. Mathematicians express all this by saying that in the first case numerator and denominator are infinities of the same order, that in the second case the numerator is an infinity of higher order than the denominator, that in the third case the numerator is an infinity of lower order than the denominator. In general given any two infinities x and y, x is said to be of the same order, a higher order, or a lower order than y according as the ratio $\frac{x}{y}$ tends to a finite quantity, infinity, or zero. If it is desired to make a still more accurate distinction some one infinity (in general the one of lowest order among those under consideration) is called an infinity of the first order and its

square, cube, ith power called infinities of the second, third, ith order. The order of any other infinity is then determined by comparison with these powers of the chosen infinity. Stated algebraicly, let z be an infinity of the first order and y any other infinity. Then y is said to be an infinity of the kth order when the limit of $\frac{y}{z^k}$ as y and z tend to infinity is a finite quantity different from zero.

APPENDIX B.

FUNCTIONALITY.

One variable is said to be a function of another when the two are so related that a change in the one produces a change in the other.* Thus the momentum of a moving body is a function both of its mass and of its velocity. Many other examples of functionality in nature may easily be given.

Any equation between two variables defines either as a function of the other. For example

$$x^2 + y^2 = 4$$

so connects x and y that no change can be made in either without affecting the other. Consequently whether we shall call x a function of y or y a function of x is a matter of purely arbitrary choice. If the equation be solved for either of the variables, the one for which it is solved is said to be an explicit function of the other. Otherwise it is called an implicit function. Thus in the example last given x is an implicit function of y, but solve the equation for x and we have

$$x = \pm\sqrt{4 - y^2}$$

which defines x as an explicit function of y.

When for any reason it becomes desirable to indicate the fact of functional relation without specifying its exact nature we use the form

$$y = f(x)$$

which is variously read y equals a function of x, y equals the f function of x, or more simply y equals the f of x. When several functions enter into the discussion they are distinguished by the use of subscripts f_1, f_2, f_3, etc., or by the use of other letters F, ϕ

*This definition of functionality is merely a working definition for present use. In some branches of the higher mathematics it becomes necessary to define more closely and to recognize distinctions which the student is not now prepared to consider.

ψ etc., in the place of f. When in any discussion the form of the function corresponding to any functional symbol, $F(x)$ for example, has been defined, it is understood that this symbol shall continue to represent this same form throughout the discussion. For example if we have

$$f(x)=x^2+2x-a$$

then

$$f(y)=y^2+2y-a$$
$$f(c)=c^2+2c-a$$
$$f(\alpha+\beta)=(\alpha+\beta)^2+2(\alpha+\beta)-a$$

APPENDIX C.

PERMISSIBLE OPERATIONS.

In reducing equations to a simple form as in article 24, the student must see to it that the operations performed do not in any way change the character of the locus. The force of this remark is most clearly set forth by some illustrations.

The equations

$$2x + y = x - 4$$

and

$$x + y + 4 = 0$$

are equivalent, i. e., represent the same locus, since every point which satisfies either one satisfies the other also.

The equations

$$x + y - \frac{7}{3} = 0$$

and

$$3x + 3y - 7 = 0$$

are equivalent since every point which satisfies either satisfies the other also.

The equations

$$x^2 + xy - 2x = 0$$

and

$$x + y - 2 = 0$$

are not equivalent since the first is satisfied by every point on the Y axis and the second is not.

The equations

$$x = y$$

and

$$x^2 = y^2$$

are not equivalent since the latter is satisfied by points for which $x = -y$ as well as by those for which $x = y$.

A little consideration will show that the modifications to which we subject our equations may be reduced to two operations: transposition, and the introduction and rejection of factors. Still further consideration will show that a transposition of terms can

in no way affect the locus, and that the introduction or rejection of a factor has no effect upon the locus, provided that the factor is of such a nature that it cannot vanish for any values of the variables. The student will do well at this time to read the articles in some standard algebra upon reversible operations. See in particular Fine's College Algebra.

APPENDIX D.

PROJECTION.

Let P be any point and CD any plane. Let Q be any other point. Draw PQ and let R be the point in which PQ extended meets CD. Then R is the projection of Q on the plane CD from P, sometimes called the center of projection. The projection of any geometric object or group of objects is the aggregate of the projections of all the points of the object or group of objects. The projection is sometimes spoken of as the shadow of the object cast on the plane CD by a light at P, and this description is satisfactory when the point P and the plane CD are on opposite sides of the object. When this is not the case, as for Q' above, it is necessary to fall back on the definition first given. If this point P removes indefinitely, the lines joining P to the various points of the object to be projected tend to parallelism, i. e., projection from an infinite distance is by parallel lines. If in addition P removes indefinitely in a direction perpendicular to the plane CD, the projecting lines tend to parallelism and perpendicularity to CD, i. e., the perpendicular projection from an infinite distance consists of the aggregate of the feet of the perpendiculars let fall from all the points of the object upon the plane of projection. In this case the projection is said to be orthogonal and is evidently the sort already presented to the student in his study of solid geometry.

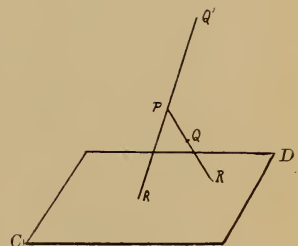

FIG. 41.

If all the points of the object to be projected are in one plane EF and the center of projection P is in the same plane it is evident that the projection will be wholly in that plane and will consist of the totality of points in which lines from P to every point of the object meet LM, the line of intersection of CD and EF. If P, remaining in the same plane, removes indefinitely in a direction

perpendicular to the line LM, the projection of the object is the totality of the feet of the perpendiculars let fall from every point of the object on LM. In this case the projection is again called orthogonal, and is evidently the sort already presented to the student in his study of plane geometry. Hereafter when no center P of projection is mentioned it is assumed that the projection is orthogonal.

If the student has a clear understanding of the preceding statements, the following propositions need no proof.

I. The projection of any length AB upon the straight line CD, which makes with AB an angle θ, is $AB \cos\theta$.

II. The projection of any contour is the sum of the projections of its component parts. Proj. $PQRS =$ Proj. $PQ +$ Proj. $QR +$ Proj. RS.

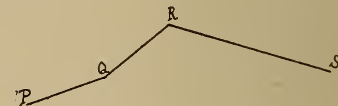

FIG. 42.

III. For purposes of projection a curved line may be regarded as the limiting form of a broken line as the number of its component parts increases indefinitely and each part tends to zero.

When the contour turns back upon itself the question at once arises whether or not the double portion shall be twice counted. So far as anything we have so far indicated is concerned the answer is yes. There is however another point of view. It is evident that if PQ be regarded as the path of a moving point, the

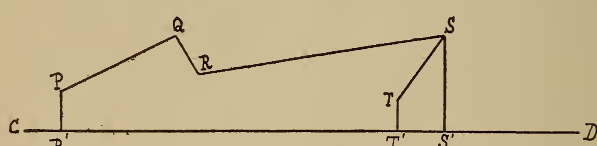

FIG. 43.

projection of PQ on CD is the amount of movement of the point in the direction CD while the point goes from P to Q. Looking at the question from this point of view, which is an important one in much mathematical work, it is evident that the total movement in the direction CD of the point which travels such a path as $PQRST$ is the sum of the projections of PQ, QR, RS, minus the projection of ST., i. e., $P'S' - T'S' = P'T'$.

The point of view just outlined is the one usually taken by mathematical writers, and in accordance with it they adopt the following conventions:

I. In considering any contour determine an initial and a terminal point.

II. The positive direction in any portion of a contour is the direction of motion of a point going from the initial to the terminal point.

III. Assume one direction or the other along the line on which projection is made as positive.

IV. The projection of any of the component lines of a contour is equal to the length of the line multiplied by the cosine of the angle between the positive direction of the projected line and the positive direction of the line on which the projection is made.

The student may now prove the following theorems:

I. The projection of any contour is equal to the algebraic sum of the projections of its component parts.

II. The projection of any closed contour is zero.

III. The projections of any two contours having the same initial and terminal points are equal.

IV. The resultant of any open contour is defined to be a line joining its initial point to its terminal point. Show that the projection of any open contour is equal to the projection of its resultant.

APPENDIX E.

IMAGINARIES.

In all work with complex numbers the student must be careful to keep in mind the fact that the term imaginary is used in a purely technical sense. In their earlier mathematical use, the terms real and imaginary undoubtedly had the significance they now have in ordinary usage, but a clearer understanding of the nature of number has come with the years and we no longer regard $\sqrt{-1}$ as imaginary in the literary sense of the word any more than we think of a fraction as a broken number.

The whole matter will perhaps be clearer if we look at the nature of the various sorts of number which are sometimes grouped together as algebraic. Algebra recognizes six operations, three direct (addition, multiplication, involution), three inverse (subtraction, division, evolution). As material to which to apply these operations the world about us presents nothing but positive integers. There are in nature no negative numbers and no fractions. An object divided into two parts becomes two objects, and it is only by imagining an undivided object to be divided, or imagining two objects to be united that we are able to talk of halves.

If now we apply any or all of the direct algebraic operations to the positive integers we get nothing new; sums, products, and positive integer powers of positive integers are all positive integers, a fact which we may express by the statement that the positive integers form a complete group out of which it is impossible to pass by means of the direct algebraic operations. If however we apply to these positive integers the inverse operation of subtraction we may as before get positive integers, but we get also a new sort of thing to which we give the name of negative integers. The equations

$$5-6=t_1, \qquad 4-6=t_2, \qquad 3-6=t_3, \qquad \text{etc.},$$

define a set of numbers t_1, t_2, t_3, of such a nature that when they are increased by 1, 2, 3, they become zero, i. e., they bear the same

relation to zero that zero bears to 1, 2, 3. It is also evident that

$$t_3 + 1 = t_2,\ t_2 + 1 = t_1,\ t_1 + 1 = 0,$$

i. e., t_3, t_2, t_1, differ from each other in the same way as any three consecutive integers. In other words the inverse operation of subtraction enables us to define a series of negative integers, each one symmetric with respect to zero to a corresponding positive integer, and forming with zero and the positive integers an unlimited sequence of numbers such that it is possible to pass from any one to those consecutive with it on either side by the addition or subtraction of unity.

Now while nature knows no such things as a negative number and laughs at the idea of continuing the process of subtraction after zero is reached, she nevertheless presents numerous illustrations of sequences of numbers arranged symmetrically with respect to some neutral point. Distance above and below sea level, north and south of the equator, or east and west of some chosen meridian; temperature above and below the freezing point; assets and liabilities, profit and loss are examples of such sequences, any one of which may be used to illustrate the properties and laws of operation connected with negative numbers.

If we continue to apply the inverse algebraic operations we derive other new numbers to which we give the names of fraction, irrational, imaginary; and then find that with positive and negative integers, fractions, irrationals, and imaginaries the field is again closed, in that no finite number of algebraic operations direct or inverse can give us anything new. For each of these new things, with the exception of the imaginary, it is possible to find with but little difficulty excellent illustrations in nature, and thereby render clearer to our minds the laws under which we work with these new symbols.

It is even possible to find in nature an illustration of imaginaries. Let us adopt the usual convention and represent positive and negative numbers by points on a straight line. Consider any positive number a and multiply it by $i\ (= \sqrt{-1})$ four times in succession, arriving thus at the numbers ia, $-a$, $-ia$, a. In other words, two multiplications by i have the same effect upon a as if it had been rotated 180 degrees on a circle centered at 0 and of radius a, while four multiplications are equivalent to a

FIG. 44.

rotation through 360 degrees. We may therefore, for purposes of illustration, regard multiplication by i as equivalent to a rotation of 90 degrees. From this point of view real and pure imaginary numbers may be represented by points on two perpendicular lines as in the diagram. A complex number, such as $a + ib$, may be represented by a point with an abscissa a and an ordinate b; and thus by utilizing all the points of the plane we can represent all values of a single complex variable. The assumptions by which this method of representation is built up may seem arbitrary or artificial, but they amount merely to a recognition of the direction as well as the distance of a point from the origin, and so differ in degree rather than in kind from the assumptions by which we represented positive and negative numbers by points on a straight line.

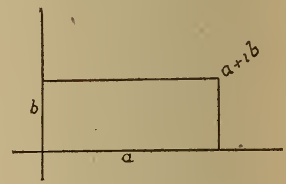

FIG. 45.

Given any complex number $a + ib$ represented by the point P, the angle θ is called the amplitude or argument of P; and ρ, always considered positive, is called the modulus of P. Evidently

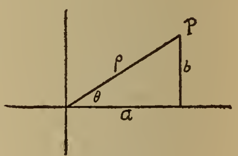

FIG. 46.

$$a = \rho \cos \theta \qquad \rho = + \sqrt{a^2 + b^2}$$

$$b = \rho \sin \theta \qquad \theta = \tan^{-1} \frac{b}{a}$$

and
$$a + ib = \rho (\cos \theta + i \sin \theta).$$

The term modulus is frequently abbreviated to mod. or denoted by a pair of vertical lines including the quantity considered. Thus

$$\text{modulus}(a + ib) = \text{mod}(a + ib) = |a + ib| = +\sqrt{a^2 + b^2}.$$

If a complex number is zero the real part and the coefficient of the imaginary part are both zero, and conversely.

If a complex number is zero its modulus is zero and conversely.

The sum of two complex numbers consists of the sum of the real parts plus i times the sum of the coefficients of the imaginary parts. This theorem may be demonstrated instantly by adding the two complex quantities in the form $a + ib$, $c + id$. Geometrically,

let the point P represent $a+ib$, and Q represent $c+id$. Draw from P a line PR equal and parallel to OQ. Then R represents $a+ib+c+id$ and

$$OR = |a+ib+c+id|.$$

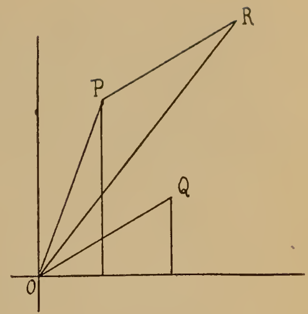

FIG. 47.

The justification of this geometric treatment of addition is left to the student. If he has ever studied physics, the idea will doubtless suggest itself to him that complex quantities might with profit be used in the discussion of such topics as the resolution and composition of forces or motions.

The geometric treatment of addition leads at once to the important theorem that the modulus of the sum of two (or any number of) complex numbers is less than, or at most equal to, the sum of the moduli. Is it possible to state a similar theorem concerning the amplitudes?

To form the product of two complex numbers $a+ib$, $c+id$ let $|a+ib|=\rho$, amp.$(a+ib)=\theta$, $|c+id|=\mu$, amp.$(c+id)=\phi$, then $(a+ib)(c+id)=\rho(\cos\theta+i\sin\theta)\,\mu(\cos\phi+i\sin\phi)$

$$=\rho\mu(\cos\theta\cos\phi-\sin\theta\sin\phi+i(\cos\theta\sin\phi+\sin\theta\cos\phi))$$
$$=\rho\mu(\cos(\theta+\phi)+i\sin(\theta+\phi))$$

It follows at once that the modulus of the product of two complex numbers is the product of the moduli, and the amplitude of the product is the sum of the amplitudes.

This last theorem leads to a simple geometric method of constructing the point which represents the product of two complex numbers. Let P represent $a+ib=\rho(\cos\theta+i\sin\theta)$ and Q represent $c+id=\mu(\cos\phi+i\sin\phi)$ and OA be the unit of measure on which the diagram is constructed. Construct the angle $\psi=\theta+\phi$ and at Q construct the angle $\alpha=OAP$. Then R, the intersection of the two lines so determined, represents $(a+ib)(c+id)$ $=\rho\mu(\cos(\theta+\phi)+i\sin(\theta+\phi))$ since its amplitude is the sum of the amplitudes and (since $\dfrac{OP}{OA}=\dfrac{OR}{OQ}$, i. e., $OR=OP.OQ$) its modulus is the product of the moduli.

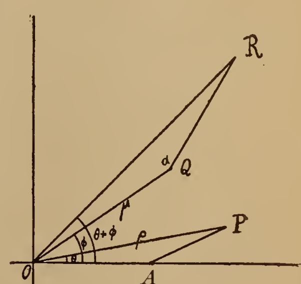

FIG. 48.

The student who desires to make a further study of this mode of representing a complex variable may read, among others, Burnside & Panton, Theory of Equations, Vol. I, Chap. XII; or Durège, Elements of the Theory of Functions of a Complex Variable, Introduction.